Tightening DGNSS Protection Levels Using Direct Position-Domain Bounding

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Abstract

We propose a novel method to compute overbounds on position errors of Differential Global Navigation Satellite System (DGNSS) solutions. Prior work has suggested that a tighter protection level bound would be possible if conservative assumptions were applied only to the position-domain error distribution model rather than to each individual range-domain error distribution model.

It has previously been unclear, however, how to ensure conservatism in the position domain, since position-domain error distributions are more geometry dependent than range-domain error distributions. This work introduces a novel approach for position-domain bounding that provides a practical means to account for geometry dependence in the error-model validation process.

The proposed process reduces conservatism used in constructing error bounds (called protection levels) and thereby has the potential to improve the availability of the DGNSS signal for safety-of-life operations.

Introduction

Protection Levels (PL) are among the most important performance criteria for safety-of-life applications of a Differential Global Navigation Satellite System (DGNSS) installation. PLs represent the confidence interval for the position estimate, with a confidence level that depends on the specific application. These confidence levels are usually derived from a model distribution for a particular probability (e.g. the 99.999999% confidence level might be computed for a Gaussian noise model).

In order to determine whether an operation can be conducted safely, the PL is compared against a reference: the Alert Limit (AL). The AL is dictated for the particular operation being executed and is part of a standard procedure. If the computed PL exceeds the required AL, the operation (which may be landing an aircraft) cannot proceed. As requirements on vertical errors are often stricter than on horizontal errors, it is useful to visualize numbers in the vertical direction. For applications related to automated landing of civil aircraft, the vertical AL is at 35 m for LPV-200 approaches and at 10 m for Category III. Should the vertical PL exceed the vertical AL, the DGNSS safety-of-life service becomes unavailable.

This paper introduces a method that allows a reduction of the PL by modeling DGNSS position errors in the position domain, as opposed to the range domain, as usually done in practice [1, 2]. In this context, the traditional method of computing PLs (which are reported in the position domain) from range-domain considerations will be referred to as...
Indirect Position-Domain Bounding or Indirect PDB; the proposed method provides PLs without regard to range-domain considerations and will therefore be referred to as Direct PDB.

The proposed direct PDB differs from indirect PDB in its treatment of the error models that underly PLs. Figure 1 offers a comparison of the two methods. The traditional method uses range error measurement data to compute overbounding (conservative) error models that are then mapped into the position domain to produce PLs. The direct method takes position errors and an overbounding model is applied directly to the data to produce PLs.

The remainder of the paper is structured as follows: First the method currently used for modeling position domain errors is outlined and contrasted with the proposed method in the next section. The following section lays the foundations of the proposed direct method and is followed by a section that looks into the nature of the measurements required for Direct Position-Domain Bounding. Following that, a section with some preliminary results shows how the method could be implemented on real data and what these data may look like. The section after that is devoted to the discussion of results and insights. The paper also has two appendices: one provides an integrity proof that supports the direct PDB method, and the other presents an outlook to the future and how this method might provide enhanced DGNSS service.

**Comparison of Direct and Indirect Position-Domain Bounding**

In order to fully appreciate the benefits of Direct Position-Domain Bounding, it is helpful to establish the differences from the traditional method. This section presents the basic concepts behind both methods and an illustrative example of how excess conservatism is reduced by describing error models directly in the position domain.

**Indirect Position-Domain Bounding**

Traditionally, error bounding has been based on producing a position-domain error model that is constructed from the modeled error on each ranging measurement; the error is modeled separately for each satellite and mapped into the position domain. Because this procedure relies on conservative range-domain error models to construct the position-domain

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**Figure 1:** Comparison of Direct and Indirect PDB. **Indirect PDB** (left branch): The range error data $E_i$ are computed for each satellite individually and modeled with a conservative overbound, e.g. as $N(0,\sigma_i)$. The resulting range-domain error model is transformed into the position domain and the components of the error models are summed in each direction (East, North, Up). **Direct PDB** (right branch): The position-domain error is computed from range error data and satellite geometry. The overbounding model is applied directly in the position domain, resulting in a different position-domain (E,N,U) error distribution.
error model, the method will henceforth be referred to as “indirect” Position Domain Bounding (Indirect PDB).

The relationship between ranging error and position-domain error is dictated by the range-to-position matrix S:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_b \end{bmatrix} = S \cdot \varepsilon_p.$$  \hfill (1)

Here \( \varepsilon_x \) is the \( 3-D \) position error, \( \varepsilon_b \) is the clock error, and \( \varepsilon_p \) is the measurement error on each pseudorange; the matrix \( S \), used in obtaining the position solution, is a weighted pseudo-inverse computed from the geometry matrix \( G \) and a weighting matrix \( W \):

$$S = (G^T \cdot W \cdot G)^{-1} \cdot G^T \cdot W.$$  \hfill (2)

The weighting matrix \( W \) depends on the elevation of each satellite \( i \).

The structure of (1) implies that the directional position errors, which are the three elements of \( \varepsilon_x \), are each constructed as the sum of \( N \) random variables, where \( N \) is the number of satellites in view. To illustrate this relationship, consider one element of \( \varepsilon_x \), the Vertical Position Error (VPE). Extracting the vertical row from (1) gives the following equation for the VPE:

$$VPE = \sum_{i=1}^{N} (S_{v,i} \cdot E_i)$$  \hfill (3)

Here \( S_{v,i} \) refers to an element in the matrix \( S \) from the vertical row and from the column corresponding to satellite \( i \). The term \( E_i \) refers to the error on that same satellite.

Indirect PDB uses equation (3), or equivalently equation (1), to construct a model of VPE by first constructing a set of models for each satellite error \((E_i)\), as shown on the left branch of Figure 1.

First, statistical range error data are obtained. Notationally, these are computed for a stationary reference antenna by subtracting the differentially corrected pseudorange from the true range (based on a calibrated antenna location). For instance, one approach for computing the instantaneous range error is:

$$E_i = [\rho_{dc,i} - \frac{1}{N} \sum \rho_{dc,i}] - [r_i - \frac{1}{N} \sum r_i]$$  \hfill (4)

In this example, \( \rho_{dc,i} \) indicates the differentially corrected pseudorange for satellite \( i \), and \( r_i \) indicates the true range for that satellite.

Because the clock offset is not known, it is subtracted out by removing the average pseudorange and range values across all satellites.

The next step, labeled Overbounding in the left branch of Figure 1, compiles a large population of \( E_i \) samples in order to model the distribution for each satellite. Models are constructed to be conservative in that the model slightly overpredicts the occurrence of large errors. These conservative range-domain probability-density function (pdf) models are often called “overbounds”. An overbar notation will be used in this document to identify overbounds. For instance, the true range error distribution \( p(E_i) \) is modeled by the overbound \( \overline{p}(E_i) \).

The next step in Indirect PDB, labeled “Range-to-Position Mapping” in the left branch of Figure 1, converts the range error overbounds into a position error overbound. Assuming errors are independent, the true position error pdf is a convolution of the range error pdfs. This result is an extension of the fact that when two random variables (e.g. \( x \) and \( y \)) are summed, the result (e.g. \( z = x + y \)) has a convolution of the tails of the position-error overbounding pdf. The next step is to compute the range-domain error overbound. Assuming errors are independent, the true position error pdf is a convolution of the range error pdfs. This result is an extension of the fact that when two random variables (e.g. \( x \) and \( y \)) are summed, the result (e.g. \( z = x + y \)) has a distribution \( h(z) \) obtained by convolution as follows:

$$h(z) = \int_{-\infty}^{\infty} g_1(x)g_2(z-x)dx.$$  \hfill (5)

Under certain conditions [4, 5], a conservative position error overbound can similarly be obtained from range-domain error bounds via repeated convolution.

Typically, range errors are modeled as zero-mean Gaussian distributions, where the Gaussian variance is a function of satellite elevation. For Gaussian range error pdfs, the position error is also Gaussian, and its variance may be computed in closed form:

$$\sigma_p = \sqrt{\sum (S_{v,i}^2 \sigma_i^2)}.$$  \hfill (6)

The final step in Indirect PDB is to construct a PL from the position-error overbounding pdf. The PL is computed by integrating the tails of the position-error overbound and determining at what magnitude of error the integrated tail probability equals a specified integrity risk. Given a Gaussian distribution, for example, the PL can be expressed simply as a multiplier times the conservative position-domain sigma:

$$PL = k \cdot \sigma_p.$$  \hfill (7)

The Error Data need to be collected over long enough periods of time, so as to support the required confidence in the models.
In the modeling step, the data are overbound by a conservative model, which could be a zero-mean Gaussian, to facilitate integrity computations.

For indirect PDB, $E_i$ is modeled separately for each satellite vehicle (SV) by collecting measurements for all satellites and overbounding the data with some probability distribution. The overbounding model of $E_i$ can then be used in the computation of an overbounding position-domain error [4, 5].

In the context of error bounding, conservative models of errors bound actual error distributions in the sense that for a given level of probability, the magnitude of the error model is bigger than what can be found in the data. “Excess” conservatism refers to models where conservative assumptions have lead to PLs that are provably larger than necessary.

Equation (3) indicates that the position-domain error is a weighted sum of range-domain errors. If those range-domain errors are independent, then their distributions can be convolved to obtain the true position-domain error distribution through repeated application of (5). Similarly, under certain conditions [4, 5], an overbounding model of the position-error distribution can be obtained by repeated convolution of overbounding range-domain error-distribution models. To see this, consider a variable $Z$, which is a random variable, such that $Z = X + Y$. Let the pdfs of $X$ and $Y$ be $g_1(x)$ and $g_2(y)$, respectively. The pdf of $Z$ will be $h(z)$, and, according to equation (5) satisfies:

$$h(z) = \int_{-\infty}^{\infty} g_1(x)g_2(z-x)dx. \quad (8)$$

Now let $\bar{f}(x)$ overbound $f(x)$, $\bar{g}(y)$ overbound $g(y)$ and

$$\bar{h}(z) = \int_{-\infty}^{\infty} \bar{f}(x)\bar{g}(z-x)dx \quad (9)$$

Using equation (9) introduces additional conservatism with each convolution. It seems plausible that, for non-Gaussian distributions, a tighter bound might be obtained working with the real position-domain error distribution - perhaps obtained by repeated convolutions of [5] - and only applying conservative modeling to the final result.

**Direct Position-Domain Bounding**

Direct PDB dispenses with the range-domain as an intermediary and, instead, constructs an overbound directly from position error samples. In other words, for a stationary reference antenna, each sample error $E_{pos}$ is computed by differencing the differentially-corrected position solution $x_{df}$ from a surveyed ground truth $x_{true}$.

$$E_{pos} = x_{df} - x_{true} \quad (10)$$

Given a population of position error samples acquired for related satellite-geometry conditions, that population can be approximated conservatively with a position-domain overbound. This process is labeled “overbounding” on the right branch of Figure [1]. The result is an overbounding model $\bar{p}(E_{pos})$ that depends on the geometry of the visible satellites, as characterized by the Geometry matrix $G$.

$$\bar{p}(E_{pos}) = f(E_{pos}; G) \quad (11)$$

Here $f$ is used to represent an arbitrary (not-necessarily Gaussian) function with the properties of a pdf (e.g. always positive, integrates to unity). Geometric dependence may reflect azimuth and elevation-dependent noise for each individual satellite as well as the effects of satellite spatial-distribution (i.e. geometric diversity) on the solution matrix $S$.

The PL can be computed in a final step, as shown in Figure [1] from the position-domain overbound. Most typically, the PL is evaluated in one dimension at a time. For instance, the Vertical PL is computed by assessing the largest vertical error (VPE) beyond which the integral of $f(E_{pos}; G)$ is equal to a specified integrity risk.

The Direct PDB approach offers certain benefits and introduces certain complications as well, as it aims at predicting the distribution of the VPE for an ever-changing satellite geometry. The process yields a reduction in excess conservatism by directly introducing a conservative model of the VPE distribution that does not require modeling in the range domain. The model is based on conservatively bounding position error data, rather than range error data, and so conservatism and inflation are only introduced for one model, instead of $N$ models.

In direct PDB there is no need to convolve different overbounding models, as there is only one model and it is applied directly to the distribution of interest.

Position-domain errors differ from range-domain errors in that they stem from a noise process with an underlying weighted sum, as in equation (3). As such, the pdf of the position-domain error can be thought of as a convolution of various pdfs. In accordance with the central limit theorem, even if the ranging errors are non-Gaussian, the resulting position-domain distribution should tend towards a Gaussian distribution as the number of ranging measurements ($N$) increases.
Comparison of an Overbound Tightness-of-Fit for a Synthetic Example

Figure 2 shows the results of a simulation experiment. A truncated exponential distribution was used as a model for a “heavy-tailed” distribution (labeled as range-domain errors in the left plot). The range-domain errors are bounded by a Gaussian model; the model has a zero mean and the smallest standard deviation \( \sigma_{RD} \) for which all points on the distribution are bounded.

![Figure 2: The effect of summing 10 heavy-tailed pdfs](image)

The range-domain overbound \( \tilde{p}(E_i) \) is a Gaussian model with zero mean and standard deviation \( \sigma_i \):

\[
\tilde{p}_{RD}(E_i) = \mathcal{N}(E_i; 0, \sigma_i).
\]

The range-domain overbound is plotted as a dashed line in the range-domain plot in Figure 2, and the standard deviation of the overbound, \( \sigma_i = 1.25\sigma_c \).

From Figure 2 it is evident that the range-domain overbound \( \tilde{p}(E_i) \) satisfies the following conditions:

\[
\tilde{p}_{RD}(E_i) > p(E_i) \forall E_i < 0 \tag{14}
\]

\[
\tilde{p}_{RD}(E_i) \leq p(E_i) \forall E_i \geq 0. \tag{15}
\]

In the position domain plot, the data are generated by convolving the range-domain data with itself ten times, which will be denoted as:

\[
p(E_x) = p^{10}(E_i), \tag{16}
\]

where \( E_x \) represents the error along one spatial dimension in the East-North-Up reference frame, and the \( *^{10} \) operator indicates a ten-fold convolution of a function with itself. The position-domain error is plotted as black crosses in the right-hand side plot of Figure 2.

The traditional position-domain overbound, given by the indirect method is computed from the ten-fold convolution of the range-domain overbound with itself:

\[
\tilde{p}_{PDB}(E_x) = \tilde{p}_{RD}(E_i)^{10} = \mathcal{N}(E_i; 0, \sqrt{10}\sigma_i), \tag{17}
\]

where \( \sqrt{10}\sigma_i = 3.953\sigma_c \) which is plotted as a dashed line in the right-hand side of Figure 2.

The result of applying a Gaussian overbound directly to the position-domain error \( E_x \) will be denoted as \( \tilde{p}_{MPDB}(E_x) \). This overbound, plotted as a solid line in Figure 2, has a smaller standard deviation \( \sigma_d \) than the indirect overbound:

\[
\tilde{p}_{MPDB}(E_x) = \mathcal{N}(E_x; 0, \sigma_d). \tag{18}
\]

The smallest \( \sigma_d \) that overbounds \( p(E_x) \) is \( \sigma_d = 3 \), which is smaller than \( \sigma_i = 3.953 \) yet provides a conservative bound for the position-domain error. The example shows that direct position-domain bounding can bring about a reduction in excess conservatism, in this case of nearly 25%; this is even possible in a scenario with errors that are known in closed form and can accurately be bounded.
**Implications of Position-Domain Bounding**

By reducing excess conservatism, PDB has the potential to enable many new applications for safety-of-life systems. One possible benefit would be improved availability for certain safety-of-life systems (e.g. precision approach and landing using augmented GPS). These operations are subject to ALs that allow landing only if the PL does not exceed the AL. Shrinking protection levels would thus enable precision operations under a greater range of conditions, particularly under degraded conditions (e.g. satellite fault or jamming conditions).

Another possible benefit might even be to enable augmentation systems to deliver new categories of service. Consider Space-Based Augmentation Systems (SBAS), for instance, which currently support LPV-200 approaches (very similar to Category I landings with a 200 ft decision altitude). If it were possible to shrink SBAS protection levels enough, it might be possible for an SBAS to also support Category III landings, proceeding all the way down to touchdown. For a more detailed description of how this might be accomplished, see Appendix 2.

**Error Distribution Binning**

The proposed approach to Direct PDB relies on computing an intermediate classification metric to cluster different error scenarios (with distinct but similar error distributions). Finding an appropriate \( f(G) \) requires processing data and studying the error distribution as a function of \( G \). A practical way of computing an error distribution over different values of \( f(G) \) is to study the statistics of all error measurements that correspond to a particular range of values of \( f(G) \). To that end, the range of possible values of \( f(G) \) must be segmented into “bins” and the statistics computed within each bin.

**Non-stationary Process**

A central challenge underlying position domain error distributions is their non-stationary nature. Error distributions depend on a number of different influences that include, but are not limited to overall constellation geometry, multipath sensitivity for individual satellites as a function of elevation, and time-varying ionosphere and troposphere delays. As such, the pdf of the position error is a function of time. Strictly speaking it is not possible to characterize such a process statistically because each data sample acquired is drawn from a different distribution.

From an engineering point of view, however, it is practical and reasonable to construct a histogram for a nonstationary process that is changing slowly enough that many data points can be acquired from very similar distributions. The difficulty in such an approach lies in the appropriate design of the number of bins in the histogram, as the design needs to reconcile a finely sampled model (which requires larger numbers of bins) with the time it takes to populate the bins (which requires smaller numbers of bins). The time to populate bins is an important consideration, as it competes with the time-variant nature of the probability distribution of the position-domain errors.

Managing this tradeoff is the key subject of this paper, where the question to address is whether there is a way to define a limited number of bins for PDB such that we can accumulate enough data in each bin to model the far tails without losing conservatism because the PDF changes too much within any given bin.

**Range-Domain Bounding**

Indirect PDB deals with the same tradeoff as described above, but range data are binned for individual satellite ranging errors. It is widely agreed that the major effect driving non-stationarity of ranging errors is Elevation. Typically errors are combined over bins of 5 to 10 degrees of elevation (e.g. 0−10, 10−20, 20−30, and so forth). Given approximately 10 elevation bins are used and given that, on average, each bin is populated with \( N \) data points, the total number of data points collected to populate the distribution model is \( 10N \). As the range-domain bounding approach is the standard method applied for validating GNSS augmentation systems, this population size of \( 10N \) data points stands as a baseline for what size of data collection campaign might be considered feasible.

**Naive Position-Domain Bounding**

A Direct PDB method would be based on binning position-domain errors, obtained from (10), rather than range domain errors, obtained from (12). One of the simplest conceivable binning schemes would be to bin the position error data by azimuth and elevation. The problem with such a scheme is the explosion in the number of bins: If the azimuth and elevation of each satellite were classified into one of 100 bins (a grid of 10 bins in each direction), and if each of 10
satellites were similarly characterized with 100 bins, then a total of $10^{20}$ bins (i.e., $100^{10}$) would be needed to characterize the geometry of entire constellation. Such an approach would require collecting data from $10^{20} \cdot N$ different geometries and they would need to be at least 300 s apart.

**Direct Position-Domain Bounding**

It is now evident that a different strategy is needed; this strategy needs to eliminate geometric effects and elevation effects. The method must identify a classification metric that identifies similar distributions. In this paper, we call the classification metric a Figure of Merit (FOM). The resulting FOM must classify data well enough to ensure that non-stationarity effects on the position error are significantly smaller than the maximum tolerable variability in the position error.

The fundamental problem will still remain: even if an excellent FOM is found, non-stationarity effects will still be present in each bin. The main focus of this paper is to obtain a conservative model for the worst-case distribution associated with a given FOM, given that we know a range of distribution will be represented in each bin.

Direct PDB proposes a conservative model for the worst possible distribution associated with a given FOM, even though each bin represents a range of different distributions.

The key assumption is that for this paper the problem is considered in a probabilistic sense (instead of statistically). This enables the study of non-stationary effects without any concerns on sampling effects (such as the noisy representation of empirical distribution tails based on limited available data). A key topic for further work is the study of statistical effects and the difficulty of inferring far tails of the position error distribution, given limited data.

The binning process yields a set of different operating conditions or regimes with associated error distributions. Under a particular FOM the regime position errors can, therefore, be bounded for that condition.

Adequate design of the bins is a crucial step in direct PDB. A binning scheme will be most useful if it provides a clear way of separating favorable error scenarios from unfavorable ones. This can be assessed graphically, using quantile-quantile plots. Plotting multiple distributions from different bins on a quantile-quantile plot allows straightforward visual assessment of whether one distribution overbounds another. Intuitively a normal probability plot represents a Gaussian data set as a straight line, where the inclination of the line is inversely proportional to the variance and the intercept is given by the mean.

**Theorem: Overbounding of Mixtures of Gaussians**

Consider the special case of a nonstationary random process for which the error distribution is zero-mean Gaussian at any instant, but for which the variance of that Gaussian distribution is continuously changing. If several data points are sampled sequentially and combined to create an empirical distribution, that empirical distribution is best modeled as a mixture of Gaussians. For an integrity application, the important detail is that the worst instantaneous distribution must be identified from the mixture of Gaussians, since integrity is based on instantaneous risk (and not average risk over time).

The following theorem states that the worst instantaneous Gaussian can be identified from a mixture of Gaussian distributions; by implication, an empirical histogram of error data for a nonstationary process can be used to identify the worst instantaneous error distribution (at least so long as the instantaneous error distribution is always Gaussian). A mathematical statement of this theorem is given below; a proof of the theorem can be found in the appendix.

**Theorem:** For a mixture of zero-mean Gaussians $M_1 ([\sigma_1, \sigma_2, \ldots, \sigma_N])$, the smallest $\sigma_{\text{bnd}}$ for which $N(0, \sigma_{\text{bnd}})$ overbounds $N(0, \sigma_N)$ also overbounds $M_1$.

**Proof** See Appendix I.

**Selection of a Figure of Merit**

The selection of the FOM to provide direct PDB is a critical design choice. An adequate FOM requires the distribution of the position domain error to be sensitive to the measurable quantities that make up the FOM. The relationship between FOM and error variability does not need to follow a particular function, but it needs to be monotone.

Once a FOM is chosen, the domain of the function must be split into different bins. The design of the binning strategy must reconcile competing interests: a large number of bins will provide more finely graded PLs, but the confidence levels will be lower.
Introducing a small number of bins, the confidence level increases, but the error model becomes more coarse.

The FOM needs to be such that the position-domain errors are well clustered. If this is the case, then the effects of non-stationary error distributions will be negligible within each bin and the position error will be quasi-stationary.

**Dilution of Precision as a Figure of Merit**

As a starting point we propose Dilution of Precision (DOP) as a FOM for the position error. Geometric DOP in particular is a promising candidate for describing the distribution of the VPE.

Intuitively, DOP can easily be seen to impact error behavior in the extremes. Excellent DOP values, for example VDOP ≤ 2 with N = 7 satellites in view, will likely have smaller errors than 4 ≤ VDOP ≤ 8, as might occur with N = 4.

Note that the FOM is not intended to predict error magnitudes, it only needs to guarantee small position errors for certain areas of operation while being sensitive to increases in error magnitude. In this sense, GDOP is a promising FOM for the VPE.

**Preliminary Results**

As a proof of concept, we applied DOP as a classification metric for a sample set of differentially-corrected positions solutions. Data were collected from the Continuously Operating Reference Station (CORS), a service provided by the National Oceanic and Atmospheric Administration (NOAA). In this setup, data were collected for the entire day of January 1, 2013, sampled at 30 s. The data for CORS station ZOA2 were used, together with the known position of the station, to compute local differential corrections; the differential corrections were applied to data from a nearby CORS station, P222, which in turn allowed for the computation of a position-domain error. Note that in order to obtain a wide range of DOP readings, the data were artificially degraded by blacking out 9 different satellites over the entire day; this explains why the DOP values climb above 5, which is extremely rare under all-in-view conditions and should not occur more than once in the unaltered dataset.

The position-domain error plotted in Figure 3 is binned as a function of Geometric DOP. The range of GDOP values is broken up into three intervals: 0 < GDOP ≤ 2, 2 < GDOP ≤ 5, and 5 < GDOP. The differential position errors are binned by GDOP and plotted in a quantile-quantile format, with Gaussians of indicated standard deviation plotted as a reference. The first observation immediately apparent from the plot is that for a GDOP below 2, the vertical position error is bounded by a Gaussian of standard deviation 2 m, and for GDOP ≤ 5 this number grows to 3 m. Upon further inspection the curve corresponding to GDOP > 5 offers an insight into how the Quantile-Quantile plot aids assessment of heavy tails. The plot is bounded by the σ = 4 m line on the right hand side, but approaches the σ = 7 m line on the left hand side of the plane. This behavior of crossing various σ-lines is an indication of a heavy-tailed distribution.

Finally, the plot in Figure 3 exemplifies how the integrity proof informs the binning process. The heaviest component in a mixture of Gaussians dominates the tails of the distribution. The heaviest tail is that which requires the largest σ to be overbound; that is: the most shallow straight line in a Quantile-Quantile plot that touches only one data point and overbounds all others represents the largest σ in the mix.
Discussion

Direct PDB is motivated by the fact that conservative range-domain error models do not guarantee conservative position-domain models. A tradeoff is required to assure the highest possible integrity and providing the highest possible availability. Integrity is usually increased by inflating error models to accommodate non-Gaussian or otherwise unknown error distributions; inflation, however, directly impacts availability.

In this sense, Direct PDB offers the possibility of accommodating non-Gaussian error models and tightening PLs without modifying the integrity risk. This observation makes Direct PDB particularly interesting for Space Based Augmentation Systems, where the difference between protection levels and actual error performance is much higher than in other navigation systems.

The preliminary results presented in the previous section confirm the suspicion that DOP is an adequate FOM for the VPE. Under nominal, all-in-view conditions the DOP will not rise above 2 very often; on the particular day and at the particular place of the sample data set it only does so for about 3.32% of the time.

The results confirm the feasibility of the proposed direct PDB method, but they also highlight some of the shortcomings of the indirect method.

Limitations of Conventional Error Bounding

The main drawback of indirect PDB is that error distributions are not known exactly; they are modeled, but conservatism is introduced at various stages of the process because models are not infinitely accurate. If ranging errors were truly Gaussian, overbounding would not be required; in that case position-domain error distributions would naturally be Gaussian. In such a case, direct and indirect PDB would yield equivalent results.

The pdf of the error distribution of a sum of independent random variables becomes the convolution of the pdfs of the two random variables. In consequence, the probability distribution of the VPE, as defined in (3) becomes a convolution of N different Gaussian distributions of zero mean and standard deviation $\sigma_i$. For this reason, Conservative RD distribution models do not ensure a conservative PD distribution model, unless certain conditions are met. These conditions are:

- The error distribution and its model must both be symmetric, unimodal, zero mean, and independent [4].
- The error distributions need only be independent, but an overbound and an underbound must be found for each distribution [4].

Infinite Tails and Position-Domain Bounds

The restrictions on overbounding models, described above, stem from the fact that the position-domain error is a sum of random variables and the indirect modeling process requires a convolution of overbounding models, as stated in equation (5). When two different functions (e.g. $g_1$ and $g_2$) are convolved with each other, signal energy from the tails of either of the two functions can be mapped into the core of the resulting distribution.

By contrast, in direct PDB the bounds are calculated by inspecting error data to the integrity limit, but no information beyond that limit is required for compliance with requirements. Unlike range-domain bounds, position-domain bounds are not used in any further computations; they are the final product that needs to be specified up to a certain limit, but not beyond. In terms of the position domain bound this means that the errors do not need to be modeled beyond the integrity limit.

Summary

The paper proposes a new method to provide DGNSS error bounds directly in the position domain, skipping the traditional range-domain bounding implemented in conventional SBAS/GBAS.

Error behavior is characterized by a Figure of Merit (FOM), where different points of operation (or bins) of the FOM are characterized separately. Thus, a real-time reading of the FOM provides a confidence interval for the position-domain error.

As part of the argument we provide proof that the proposed binning scheme guarantees error bounding for adjacent bins, as long as the errors can be modeled as instantaneous Gaussians of varying $\sigma$. 
Since data in one bin bound the data in all preceding bins, the method provides a way of introducing inflation.

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Appendix I

This appendix was added to prove that:

**Theorem:** For a mixture of zero-mean Gaussians \( M(\sigma_1, \sigma_2, \ldots, \sigma_N) \), the smallest \( \sigma_{bnd} \) for which \( N(0, \sigma_{bnd}) \) overbounds \( N(0, \sigma_N) \) also overbounds \( M \).

Throughout the proof, the following notation convention will be observed:

1. A Gaussian probability distribution \( N(\mu, \sigma) \) with mean \( \mu \) and standard deviation \( \sigma \) will have an associated probability density function (pdf) \( p(x; \mu, \sigma) \) and a cumulative distribution function (CDF) \( \Phi(x; \mu, \sigma) \).

2. The argument of a pdf or CDF will be referred to as “error”.

3. A mixture of \( N \) zero-mean Gaussians will be denoted as \( M(\sigma_1, \sigma_2, \ldots, \sigma_N) \).

**Definition 1:** A CDF with zero mean \( \Phi_2(x) \) overbounds another zero-mean CDF \( \Phi_1(x) \) if for any given cumulative probability \( T \), the magnitude of the associated error \( |x_2| \) corresponding to \( \Phi_2 \) is larger than or equal to that corresponding to \( \Phi_1 \).

\[
T = \Phi_2(x_2) = \Phi_1(x_1) \iff |x_2| \geq |x_1|
\]

**Property 1:** A zero-mean CDF \( \Phi_2 \) overbounds another zero-mean CDF \( \Phi_1 \) if they satisfy:

\[
\Phi_2 < \Phi_1 \forall x > 0
\]

\[
\Phi_2 > \Phi_1 \forall x < 0
\]

For any zero-mean CDF: \( \Phi(0) = \frac{1}{2} \). CDFs are monotonically increasing functions, as their derivatives are by definition greater than or equal to zero. For any \( x > 0 \) to have a \( T_2 = P_2(x_2) \) below \( T_1 = P_1(x_2) \) requires that \( P_1(x_1) = T_2 \) be satisfied at \( x_1 < x_2 \).

**Proof:** Let \( T_2 \) be a cumulative probability, which the CDF \( \Phi_2 \) satisfies at \( x = x_2 \); thus: \( T_2 = \Phi_2(x_2) \).

Similarly, let \( x_1 \) be the point for which \( T_2 = \Phi_1(x_1) \).

If \( |x_2| > |x_1| \), then \( \Phi_2 \) overbounds \( \Phi_1 \), by definition 1. But if \( |x_2| > |x_1| \), then at \( x_2 \), due to the monotonicity of CDFs the probability value of \( \Phi_1 \) must be greater than or equal to \( T_2 \) if \( x_2 > 0 \) and less than or equal to \( T_2 \) if \( x_2 < 0 \).

- **Lemma 1:** If \( \sigma_2 > \sigma_1 > 0 \), the normal CDF \( N(0, \sigma_2) \) overbounds \( N(0, \sigma_1) \).

**Proof:** For any zero-mean Gaussian, the CDF at zero error is: \( \Phi(0; 0, \sigma) = 0.5 \forall \sigma \in \mathbb{R}\sigma > 0 \).

For any other error value \( x_0 \), the dependence of the CDF value on \( \sigma \) is:

\[
\frac{\partial}{\partial \sigma} [\Phi(x_0; 0, \sigma)] = \frac{\partial}{\partial \sigma} \left[ \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \right] =
\]

\[
= \frac{\partial}{\partial \sigma} \left[ 1 - \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \right] =
\]

\[
= \frac{\partial}{\partial \sigma} \left[ -\frac{1}{2} \text{erf} \left( \frac{x}{\sqrt{2\sigma^2}} \right) \right] \Big|_{x=x_0} =
\]

\[
= \frac{\partial}{\partial \sigma} \left[ -\frac{1}{2} \text{erf} \left( \frac{x_0}{\sqrt{2\sigma^2}} \right) + \frac{1}{2} \text{erf} \left( \frac{-x_0}{\sqrt{2\sigma^2}} \right) \right]
\]

Where \( \text{erf}(a) = -\text{erf}(-a) \) and so:

\[
\frac{\partial}{\partial \sigma} [\Phi(x_0; 0, \sigma)] = \frac{\partial}{\partial \sigma} \left[ -\frac{\text{erf} \left( \frac{x_0}{\sqrt{2\sigma^2}} \right)}{2} \right] = \frac{\sqrt{2} x_0 e^{-\frac{x_0^2}{2\sigma^2}}} {2\sigma^2}.
\]

Since the dependence of \( \Phi(x_0; 0, \sigma) \) on \( \sigma \) is always positive for \( x_0 > 0 \) and negative for \( x_0 < 0 \), it follows that:

\[
\Phi(x_0; 0, \sigma_2) > \Phi(x_0; 0, \sigma_1) \forall x_0 > 0
\]

\[
\Phi(x_0; 0, \sigma_2) < \Phi(x_0; 0, \sigma_1) \forall x_0 < 0
\]

- **Theorem:** For a mixture of zero-mean Gaussians \( M(\sigma_1, \sigma_2, \ldots, \sigma_N) \), the smallest \( \sigma_{bnd} \) for which \( N(0, \sigma_{bnd}) \) overbounds \( N(0, \sigma_N) \) also overbounds \( M \).

**Proof:** \( M \) is a weighted mean and no individual weight exceeds unity (\( \frac{1}{M} \leq 1 \)). This means that for any error \( x \), the CDF is
Figure 4: Sketch of points used in the proof of property 1.

\[ \Phi(x) = \sum_{i=1}^{N} L_i M_i \Phi(x; 0, \sigma_i) \leq \Phi(x; 0, \sigma_N). \]
Therefore any Gaussian that overbounds the heaviest component \( N(0, \sigma_N) \) of the mixture \( M_1 \) also overbounds \( M_1 \).

### Appendix II

The corrections computed by operational space-based augmentation systems (SBAS) consistently achieve a high quality \[8\] that, the authors believe, has the potential to support category III (Cat-III) operations without requiring a local ground facility. We envision a system that would use differential corrections computed by SBAS, but with enhanced fault detection provided by specialized autonomous monitors (such as \[9\]) and collaborative monitors (such as \[10\]). If feasible, such a system would have two major advantages over ground-based augmentation systems (GBAS): It would reduce the installation costs generally associated with GBAS, and it would be more robust to radio-frequency interference (RFI) than GBAS.

The approach proposed in this paper would make due without a local ground facility. In this sense, the proposed approach is different from the notion of a Local Airport Monitor (LAM), which has previously been explored as a means to leverage SBAS corrections for precision landing applications \[11\]. In LAM, SBAS corrections were processed at a local ground station, checked for integrity with local ground-based monitors, and re-broadcast to users via the VDB. Instead, in the proposed system users would use SBAS-computed differential corrections and perform their own integrity checks on board, autonomously for fast faults, or via networked collaboration for slow faults.

The challenges involved in supporting Cat-III service with SBAS differential correction stem from two requirements: Time-to-Alert (TTA) and Vertical Alert Limit (VAL). SBAS currently provides LPV-200 service, which operates with a TTA greater than 6 s, while Cat-III requires a 2 s time-to-alert. The Vertical Alert Limit for LPV-200 service is 35 m; by contrast, Cat-III service requires a VAL of 10 m. Availability would be abysmal if SBAS were to be used for Cat-III approaches without modification, as typical SBAS protection levels very often exceed 10 m. This paper will introduce an alternative concept for obtaining a vertical protection level (VPL) that is more compatible with Cat-III operations.

Our approach is based on two capabilities: local monitoring for offnominal events and tight bounding of nominal errors. Local monitoring is essential to meet TTA requirements and to achieve higher spatial resolution (e.g. for ionosphere anomaly detection) than is possible with SBAS alone. Presuming that offnominal events can be excluded through supplemental monitoring, a much tighter protection level for the SBAS-corrected position solution is possible, particularly if the position-bound is formulated directly in the position domain. (By contrast, typical protection levels are based on root-sum-squared range-domain error bounds, an approach which introduces significant excess conservatism).

The notion of achieving Cat-III using SBAS corrections is not unreasonable given actual SBAS performance under nominal conditions. According to \[8\], WAAS is able to provide a service with errors below 2 m 95% of the time. Assuming a truly Gaussian distribution and extrapolating out to a \(10^{-9}\) integrity bound implies a protection level of approximately 6 m bound. Thus, margin exists relative to a 10 m VAL to cover inflation needed, for instance, to cover heavy distribution tails \[12\]. Though supported by operational SBAS data, the notion of satisfying a 10 m VAL seems very aggressive relative to the protection levels provided by current SBAS systems. In effect, our proposed method would remove conservatism in existing protection level calculations needed for the following reasons.

- To account for a long time-to-alert: SBAS broadcasts sigmas are inflated in part to provide integrity given a relatively long time-to-alert of 10 seconds. A significant reduction in TTA is required for Cat-III operations and will reduce the potential for faults to develop over time before detection. We contend that fast time-to-alert may be possible using aircraft based fault monitoring methods, such as those discussed in \[9\].
• To account for ionosphere undersampling: Another reason broadcast SBAS error models are inflated is to account for uncertainty in the ionosphere correction model, which must be interpolated between nodes spaced regularly on a lat-lon grid. Conservatism in SBAS broadcast sigmas, to account for this uncertainty can be mitigated, potentially, by collaborative aircraft monitoring. For instance, a networked ionosphere monitor can provide sensitivity to anomalies on a smaller scale than SBAS monitoring networks [10].

• To account for range-domain over conservatism: Yet another reason why SBAS broadcast error models are conservative is that they have been defined by Indirect PDB methods. As discussed in this paper, direct PDB methods have significant potential to reduce conservatism in order to obtain tighter protection levels (and potentially higher availability).

In short, with the addition of appropriate monitoring and formal bounding methods, it seems possible that SBAS might provide Cat-III service with high availability. A detailed feasibility assessment will require extending theories presented in this paper and the other papers presented above; also, a large data collection effort would be required. Though the effort would be large, the potential benefit of our concept is enticing: Cat-III service on a continental scale without requiring GAST-D ground facilities. This would represent a dramatic reduction in the costs involved in providing Cat-III service to airports within SBAS coverage.

References


