Simultaneous Position Estimation & Ambiguity Resolution (SPEAR)

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RUNNING Title:
SPEAR: Position Estimation w/ Ambiguity Resolution

Financial Support:
NASA Safe and Efficient Surface Operations (Contract 11-1-A3.01-8393 ARC)
ABSTRACT

In typical high-precision GNSS, ambiguity resolution is performed once for each integer or combination. This paper considers an alternative class of algorithms that would continuously resolve integers through a process called simultaneous position estimation and ambiguity resolution (SPEAR). To demonstrate this concept, a novel algorithm is introduced that implements SPEAR as a Bayesian filter where integers are a deterministic function of the estimated receiver states. The proposed approach is distinctive in that it decomposes the carrier-wave measurement into two components: an apparent integer difference, used for Bayesian prediction, and a fractional phase, used for Bayesian correction. The proposed algorithm is computationally intensive, but offers potential benefits. First, the algorithm does not require continuous carrier-phase tracking and hence provides robustness to signal blockage, for example, in rough terrain. Second, because the algorithm applies integer constraints at every epoch, it converges to high-accuracy in fewer epochs than possible using conventional, unconstrained estimation.

INTRODUCTION

High-precision GNSS algorithms based on carrier-phase processing currently rely on integer ambiguity resolution to achieve centimeter-level accuracy [1-3]. Ambiguity resolution is a process that estimates the integer number of carrier wavelengths in the path length (or differential path length) to each satellite at a given instant in time. Conventional algorithms treat ambiguity resolution as a discrete transition (sometimes called integer fixing) that occurs once for each integer or integer combination. The liability of integer fixing algorithms is that they require continuous carrier-phase tracking and so are vulnerable to signal interruptions caused, for instance, when a receiver loses line-of-site to one or more satellites. Though such interruptions are rare in open environments (e.g. in many farming applications), new approaches are motivated to provide robust navigation in harsh environments where carrier-tracking is frequently interrupted (e.g. in urban canyons [4] or in the ramp-area of an airport [5]).

Accordingly, this paper investigates a class of navigation algorithms that seeks robustness to occasional signal-tracking interruptions by treating ambiguity resolution not as a discrete transition, but rather as a continuous process that happens at every epoch. This class of navigation algorithms is dubbed Simultaneous Position Estimation and Ambiguity Resolution (SPEAR) algorithms. The term SPEAR implies an integration of ambiguity resolution directly into the navigation solution. By contrast with integer-fixing algorithms, SPEAR algorithms do not rely on real-valued integer estimates to perform a one-
time ambiguity resolution; instead, SPEAR algorithms are envisioned to handle ambiguities continuously, as a constraint on the position estimation process. Thus, SPEAR algorithms should not require continuous carrier-phase tracking.

The theoretical basis for unifying ambiguity resolution and position estimation is the following. If true position and clock bias are known for a receiver, then the number of integer carrier wavelengths along the path between the receiver and satellite should be known. In other words, the integer vector $\mathbf{N}$ at a particular time is an algebraic function of the true receiver state vector $\mathbf{x}$:

$$\mathbf{N} = f(\mathbf{x}) .$$  \hspace{1cm} (1)

Hence, at least in theory, it should not be necessary to “resolve” integer ambiguities, as (1) implies that integers should be a deterministic function of any receiver state estimate $\hat{\mathbf{x}}$.

The particular approach to SPEAR proposed in this paper is to use a Bayesian estimation framework to maintain a distribution of possible state values $\hat{\mathbf{x}}$, where each state estimate is algebraically associated with an integer. In this sense, integers are not obtained separately from the state estimate, but rather impose constraint relationships on the state estimate. This constraint relationship is tied to the changes in the carrier-phase measurements between successive epochs. As such, a critical characteristic of the proposed algorithm is that it decomposes the carrier-phase measurement into two components, an apparent integer difference component used to constrain Bayesian state propagation and a fractional phase component used in the Bayesian measurement update.

The remainder of this paper describes the new algorithm in detail. First, carrier-wave measurement models are provided, both for conventional accumulated Doppler measurements and for the derived measurements employed by the proposed Bayesian estimation algorithm (namely for the apparent integer difference and fractional phase). As background, the next section briefly reviews least-squares-based integer estimation, a process which is often employed as a pre-filter for integer ambiguity resolution. Subsequently, the Bayesian state-estimation algorithm that is the main focus of this paper is presented. To assess performance, an analysis section discusses simulations that apply the conventional least-squares and novel Bayesian algorithms to synthetic data. A brief summary concludes the paper.
CARRIER-WAVE MEASUREMENT MODELS

Most current day GNSS receivers track carrier phase using a phase-locked loop (PLL). PLL tracking precision is on the order of 10° one-sigma, which corresponds to a tracking error on the order of 1 cm for the 19 cm wavelength of the L1 frequency [6]. This precision is nearly two orders of magnitude better than that for conventional GPS code tracking, which delivers closer to 1 m accuracy (one sigma, for differential GNSS).

The caveat for carrier-phase processing is that only the phase along the wavelength is truly measured; the integer number of wavelengths along the line-of-site between the receiver and any GNSS satellite is completely ambiguous. In this sense, the carrier-phase measurement is precise but not accurate. Resolving the integer ambiguity is possible but time consuming, because ambiguity resolution requires measurement filtering over many epochs. To support precision landing on an aircraft carrier or airborne refueling, for example, five or more minutes of filtering may be necessary [7-9].

This section discusses models for measurements obtained from carrier-phase tracking. Collectively, these measurements are labeled carrier-wave measurements. The most widely studied carrier-wave measurement is obtained by integrating carrier phase changes over time. This integral is sometimes called the accumulated Doppler measurement.

In this paper, the variable $\phi$ will be used to identify the vector of accumulated Doppler measurements for all tracked satellites. The accumulated Doppler $\phi$ is a function of the receiver clock bias $b$, the vector range to each satellite $r$, the integer ambiguity vector for all satellites $N$, and the measurement error vector $\varepsilon_\phi$ [6]. Here the error vector $\varepsilon_\phi$ combines random receiver noise and multipath with systematic errors, such as troposphere and ionosphere effects. By convention, the receiver clock bias $b$ is defined in units of meters (where time is converted to length through multiplication with the speed of light).

$$\phi = r(p) + b + \lambda N(p) + \varepsilon_\phi$$ (2)

The unknown receiver position $p$ is expressly included in the above equation to emphasize an important fact: that both the range $r$ and the integer ambiguity $N$ are dependent on the unknown receiver position $p$ (at least at an initial time step). Although the dependence of the integer ambiguities $N$ on receiver position $p$ is typically neglected to simplify the position solution algorithm, this paper will explicitly consider the constraints relating the range and integer vectors, $r$ and $N$. 


In this paper, the accumulated Doppler signal is decomposed to obtain two related carrier-wave measurements: \textit{fractional phase}, denoted by the vector $\psi$, and the \textit{apparent integer}, denoted by the vector $L$. These signals are related to the accumulated Doppler signal at any epoch $k$ by the following equation:

$$\phi_k = \psi_k + \lambda \left(L_k + N\right). \quad (3)$$

The fractional phase $\psi$ is the instantaneous phase of the arriving carrier signal. This phase might be defined as an angle (between 0 and $2\pi$ radians); for convenience, however, this paper defines the fractional phase with units of length (between 0 and the carrier wavelength $\lambda$). A model for the fractional phase can be obtained by applying the modulus function to calculate the remainder of the accumulated Doppler measurement after dividing by wavelength.

$$\psi_k = \text{mod}(\phi_k, \lambda) = \text{mod}(r(p_k) + b_k + e_{\phi,k}, \lambda) \quad (4)$$

By projecting onto the range $[0, \lambda)$, the above modulus function removes the integer, such that the fractional phase is inherently unambiguous.

The apparent integer is the complement to fractional phase. More specifically, the apparent integer is the number of phase boundary transitions that have occurred since carrier tracking began for a particular satellite.

$$L_k = \text{floor} \left( \frac{\phi_k}{\lambda} \right) \quad (5)$$

A negative apparent integer implies the zero boundary has been passed at least once; a positive apparent integer implies the one-wavelength boundary (i.e. the $2\pi$ radian boundary) has been passed at least once. In the absence of cycle slips, the apparent integer is related to the receiver states by the following equivalent model:

$$L_k = \text{floor} \left( \frac{r(p_k) + b_k + e_{\phi,k}}{\lambda} \right). \quad (6)$$
Examining this model, it is clear that the apparent integer $L$ is not precisely the same as the true integer $N$. One difference is that, while $N$ is deterministic, $L$ is a random integer variable that depends on the noise term $\varepsilon_{\phi,k}$. Even a small random noise contribution can shift the value of the integer $L$ if the argument of the floor function is near an integer boundary.

Because $L$ depends on $b$ and $p$, according to (6), the value of $L$ is ambiguous when the receiver states are unknown. This ambiguity can be eliminated by defining a related signal, the apparent integer difference, which is a difference between $L$ values at different epochs (as measured by the same receiver for the same satellites). The following equation defines the apparent integer difference $M$.

$$ M_k = L_k - L_{k-1} \quad (7) $$

The apparent integer difference $M$ can be related to the receiver states by substituting (6) into the above definition. The result is unambiguous:

$$ M_k = \text{floor} \left( \frac{r(p_k) + b_k + \varepsilon_{\phi,k}}{\lambda} \right) - \text{floor} \left( \frac{r(p_{k-1}) + b_{k-1} + \varepsilon_{\phi,k-1}}{\lambda} \right). \quad (8) $$

Instead of processing accumulated Doppler, the proposed Bayesian algorithm instead processes two derived carrier-wave measurements: the fractional phase and the apparent integer difference. The fact that both of these derived carrier-wave measurements are unambiguous simplifies algorithm implementation.

A final detail relevant to modeling carrier-wave measurements is that the proposed Bayesian algorithm assumes differential corrections (relative to a fixed base station) are used to remove systematic biases from the fractional phase and apparent integer difference measurements. For clarity, the notation $\delta$ will be introduced to indicate a differentially corrected quantity. For instance, differentially corrected accumulated Doppler measurements are labeled $\delta\phi$.

In the case of differential corrections over reasonably short baselines (order tens of kilometers), it is reasonable to assume the differential range between the reference station and the user receiver $\delta r$ can be linearized in terms of a geometry matrix $G$ [6].

$$ \delta r = G \delta x \quad (9) $$
Here the differential state vector concatenates the relative user-to-reference-station position vector and clock bias:
\[ \delta \mathbf{x} = \left[ \begin{array}{c} \delta \mathbf{p}^T \\ \delta b \end{array} \right]^T. \] Applying the above linearization to (4), the differential fractional phase \( \delta \psi \) can be related directly to the relative state vector \( \delta \mathbf{x} \).

\[ \delta \psi_k = \text{mod} \left( G_k \delta \mathbf{x}_k + \varepsilon_{\delta \phi, k}, \lambda \right) \]  \hspace{1cm} (10)

Similarly, the differential apparent integer difference can be modeled as

\[ \delta \mathbf{M}_k = \text{floor} \left( \frac{G_k \delta \mathbf{x}_k + \varepsilon_{\delta \phi, k}}{\lambda} \right) - \text{floor} \left( \frac{G_{k-1} \delta \mathbf{x}_{k-1} + \varepsilon_{\delta \phi, k-1}}{\lambda} \right). \]  \hspace{1cm} (11)

Here the differentially-corrected error vector \( \varepsilon_{\delta \phi} \) is modeled as zero mean, reflecting the removal of ionospheric, tropospheric and other systematic biases.

THE FLOAT SOLUTION

To provide a point of comparison for the Bayesian estimator, it is useful to first review the so-called float solution, which is a conventional least-squares-based process for estimating a real valued approximation of the integer vector \( \mathbf{N} \). The designation float solution emphasizes that the estimates are not truly integers (but rather are floating point values used by digital computers to represent real numbers).

The float solution operates by fusing together differential code and accumulated Doppler measurements, labeled \( \delta \mathbf{p} \) and \( \delta \phi \), to obtain a differential receiver-state vector \( \delta \mathbf{x}_k \) and the relative integer vector \( \delta \mathbf{N} \). Thus, in some sense, the float solution might be considered to be primarily a position estimation algorithm, which just happens to produce real-valued integer estimates as a byproduct. These integer estimates are also useful, however, in pre-filtering prior to integer fixing. Eventually, if the linear estimator combines enough data points, the real-valued integer estimates become sufficiently accurate that integer fixing is possible with a high degree of confidence.
Though the float solution is typically implemented as a recursive least-squares (or a Kalman Filtering) algorithm, for illustration purposes it is useful to model the float solution as an equivalent batch solution, one which processes all measurements simultaneously.

\[
\begin{bmatrix}
\delta p_0 \\
\delta \phi_0 \\
\vdots \\
\delta p_k \\
\delta \phi_k
\end{bmatrix} =
\begin{bmatrix}
G_0 & \cdots & 0 & 0 \\
G_0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & G_k & 0 \\
0 & \cdots & G_k & 1
\end{bmatrix}
\begin{bmatrix}
\delta x_0 \\
\delta x_k \\
\vdots \\
\delta N_k
\end{bmatrix}
+ \begin{bmatrix}
\epsilon_{\delta p,0} \\
\epsilon_{\delta \phi,0} \\
\vdots \\
\epsilon_{\delta \phi,k}
\end{bmatrix}
\tag{12}
\]

So long as continuous tracking is maintained, the differential integer vector is a constant. Hence, the estimator can effectively difference the code and carrier measurements at each epoch and thereby obtain a new integer observable, somewhat corrupted by carrier and code measurement noise vectors, \(\epsilon_{\delta p}\) and \(\epsilon_{\delta \phi}\). The key to obtaining this observable is that the differential code measurement \(\delta p\), modeled below, is unambiguous.

\[
\delta p_k = G_k \delta x_k + \epsilon_{\delta \phi,k}.
\tag{13}
\]

The code measurement errors of \(\epsilon_{\delta \phi}\) are large (meter-level). However, by averaging over time, the algorithm attenuates measurement noise and provides an integer estimate that is increasingly accurate.

**BAYESIAN FUSION OF CODE AND CARRIER-PHASE MEASUREMENTS**

In this section, a novel approach is proposed for integrating GNSS carrier and code measurements. The proposed algorithm is a Bayesian estimator [10], a form of estimation which explicitly models the Probability Density Function (PDF) for the state estimate. Bayesian estimators consist of two steps, which are implemented iteratively: a state propagation (or prediction) step and a measurement update (or correction) step. The state propagation maps the past state-estimate PDF to the current epoch. The measurement update then weights regions of this PDF based on how well they match sensor data. By applying such predictions and corrections iteratively through time, the PDF can be continually updated based on newly acquired measurement data.
The central concept of the proposed Bayesian estimator is that integer constraints are imposed by the apparent integer difference measurement and that these constraints can be embedded in Bayesian state propagation. Complementary measurements (code pseudoranges and carrier-wave fractional phase) are used in the Bayesian measurement update. Details are described below, following a brief discussion of nonparametric PDF modeling.

**Histogram PDF Model**

A characteristic of Bayesian estimators is that they are capable of modeling state-estimate distributions in a general manner. As such, state variables need not be assumed Gaussian distributed, as is typically assumed by least-squares estimators. This characteristic is important for SPEAR, as modeling integer constraints results in a highly non-Gaussian error distribution.

In the Bayesian estimation community, two of the most widely studied non-parametric models used to represent general PDFs are the histogram and particle models. The histogram model uses a well-defined, regular grid to represent the region of interest of the state-estimate PDF. By contrast, the particle approach uses an irregular, randomly selected cluster of points (i.e. a Monte Carlo simulation) to model the state-estimate PDF.

Of the two approaches, the particle approach is much more widely used, as the particle approach greatly reduces computational overhead by efficiently redistributing particles toward the densest regions of the PDF (i.e. the peaks). The histogram approach has started to become practical in some recent high-precision estimation applications [11], however, in part because of the increasingly low cost of parallel computation.

For high-precision navigation applications, particle models have significant liabilities [12]. One of the most significant liabilities of particle models of state error is that particle locations are determined stochastically (based on a random number generator), such that the results of processing are not repeatable, even when particle models are applied twice to the same measurement set. Another liability of particle models is that they are ill-suited for high integrity navigation applications where capturing low-density (i.e. far tail) regions is essential, such as in safety critical aviation applications. This liability is related to the very characteristics that make particle filters computationally efficient (e.g. the fact that intermittent resampling drives particles away from low-density regions toward high-density regions of the PDF).

Given these trade-offs, a histogram model is used to implement the proposed Bayesian algorithm. This histogram model consists of probabilities distributed on a regular grid across the four-dimensional receiver state space (three positions and...
time). The resulting number of grid cells, and the corresponding computational cost of the algorithm, is relatively high. In
the analysis described later in this paper, for example, a grid of 1.6 billion nodes was used.

Measurement Update Step

In a Bayesian estimator, the measurement update (or correction) step obtains a new (or posterior) model of the state PDF
by weighting an old (or prior) PDF model based on currently available sensor data. In the initial time step, the prior model
might simply be a uniform distribution over the state space (as initially no information exists to localize the user receiver). In
subsequent time steps, the state PDF becomes better and better resolved, allowing for the receiver state to be estimated with
progressively better precision.

A generic equation for the measurement update is obtained from Bayes Rule. Here $\delta y$ is the vector of differentially
corrected sensor measurements and $\delta \hat{x}$ is the relative-receiver state estimate. Unlike the sensor vector, the state estimate is
not a deterministic value, but rather a distribution of possible values as defined by the PDF below.

$$p(\delta \hat{x}_k | \delta y_k) = C p(\delta y_k | \delta \hat{x}_k) p(\delta \hat{x}_k)$$  \hspace{1cm} (14)

This equation weights the prior distribution by a sensor error model, which takes into account the values of the sensor
measurements $y_k$ at the current epoch $k$. The spatially uniform scaling factor $C$ is selected to ensure that the resulting
posterior $p(\hat{x}_k | y_k)$ integrates to unity probability, as required for a PDF.

The measurement update equation, equation (14), relies on the existence of a reasonable model for the sensor
measurement error distribution, $p(y_k | \hat{x}_k)$. In this work, we assume Gaussian models for sensor measurement error:

$$p(\delta y_k | \delta \hat{x}_k) = \frac{1}{(2\pi R)^{1/2}} \exp\left(-\frac{1}{2} \hat{\epsilon}^T R^{-1} \hat{\epsilon}\right),$$  \hspace{1cm} (15)

where $R$ is the measurement-error covariance and where $\hat{\epsilon}$ is the error estimate, which is the difference between the actual
measurement $y_k$ and to the expected measurement $h(\hat{x}_k)$ for each value of the state estimate $\hat{x}_k$. 

10
\[ \hat{\varepsilon}_k = \delta y_k - h(\delta \hat{x}_k) \quad (16) \]

The error estimate vector \( \hat{\varepsilon} \) includes errors for all code-phase and carrier-wave fractional phase measurements.

\[ \hat{\varepsilon}_k = \begin{bmatrix} \hat{\varepsilon}_{\delta r,k} \\ \hat{\varepsilon}_{\delta \phi,k} \end{bmatrix} \quad (17) \]

The code-phase error estimate is computed for each grid cell in the state-estimate histogram by rearranging (13).

\[ \hat{\varepsilon}_{\delta r,k} = \delta \rho_k - G_k \delta \hat{x}_k \quad (18) \]

Notionally, the carrier-phase error for each grid cell might be computed likewise by rearranging (10).

\[ \hat{\varepsilon}_{\delta \phi,k} = \text{mod} (\delta \psi_k - G_k \delta \hat{x}_k , \lambda) \quad (19) \]

This expression is inconvenient, however, as it results in a carrier-phase error distributed on the range \([0, \lambda]\), with probability densest near the edges of the domain and sparsest in the middle. To simplify processing and maintain a unimodal distribution, it is convenient to instead reformulate the error model as follows, such that the phase-tracking error lies on the range \([-\frac{1}{2} \lambda, \frac{1}{2} \lambda]\).

\[ \hat{\varepsilon}_{\delta \phi,k} = \delta \psi_k - G_k \delta \hat{x}_k - \lambda \text{round} \left( \frac{\delta \psi_k - G_k \delta \hat{x}_k}{\lambda} \right) \quad (20) \]

The resulting error distribution is unimodal and truncated over a one-wavelength range. The error may be modeled as approximately Gaussian if the standard deviation of the carrier-phase error is sufficiently small. For example, consider an approximately Gaussian carrier phase error distribution with standard deviation of 1 cm that is truncated at a half-wavelength (at +/-9.5 cm for the GPS L1 frequency). The truncation occurs at a distance of approximately 9.5\(\sigma\) from the center of the distribution, so for all practical purposes, the truncation effect can be neglected.
State Propagation Step

The correction step, defined by (14), assumes that a prior distribution is available for the state estimate at the current epoch \( k \). In general, this prior must be constructed by propagating forward the state estimate from the previous time step \( k-1 \). This process is called the state propagation (or prediction) step in Bayesian estimation.

State propagation is typically performed using a physics model (e.g. a kinematic motion model) or inertial measurements (e.g. accelerometer and gyroscope data) to relate GNSS data from one time step to another. Typically, sensing or modeling errors blur the PDF during propagation. This blurring effect can be computed by the following predictor equation, which applies generally to all Bayesian estimators.

\[
p(\hat{x}_k) = \iiint p(\hat{x}_k | \hat{x}_{k-1}) p(\hat{x}_{k-1}) dV
\]

(21)

The variable \( dV \) indicates the volumetric element of integration in the state-estimate space.

In effect, the above equation represents a convolution, where the old posterior distribution (from epoch \( k-1 \)) is blurred through convolution with a propagation-error kernel to obtain a new prior distribution (for epoch \( k \)). In the above equation, both the prior and the posterior are conditioned on all sensor data up to the previous time step (through \( y_{k-1} \)). By convention, this conditionality is omitted from the above equation for compactness.

The Bayesian estimation method proposed in this paper has a state-propagation step that is unusual for two reasons. First, the propagation step operates not on the state directly, but rather on a function of the state: the estimated apparent integer difference \( \delta L(\hat{x}) \). Second, the propagation step is deterministic. The update is deterministic because the apparent integer difference \( \delta M_k \) precisely relates the estimated apparent integers between two different epochs. That is:

\[
\delta M_k = \delta L_k - \delta L_{k-1}.
\]

(22)

The propagation is deterministic because each grid cell for the state-estimate (both at epoch \( k \) and \( k-1 \)) is associated with a unique error estimate \( \hat{e}_{\phi,k} \), computed using (20). Because the apparent integer difference \( \delta M \) introduces no additional
random error, the mapping between the integers associated with any one grid cell from epoch \( k-1 \) and any grid cell at epoch \( k \) is deterministic. Quantifying this effect, the state propagation model (21) can be reformulated as follows.

\[
p_{-}(\delta L(\hat{x}_k)) = \iint p(\delta M_k)p_{+}(\delta L(\hat{x}_{k-1}))dV
\]

(23)

Here the minus subscript in \( p_{-} \) is added to emphasize that the output distribution is a prior PDF for the current epoch; similarly, the plus subscript of \( p_{+} \) is added to emphasize that the input distribution is a posterior PDF for the prior epoch.

Because the value of \( \delta M \) introduces no additional randomness, its distribution is a unit impulse. Hence, substituting the unit impulse at \( \delta M \) for the measurement model in (23) and applying identity (22) gives:

\[
p_{-}(\delta L(\delta \hat{x}_k)) = p_{+}(\delta L(\delta \hat{x}_k) - \delta M_k).
\]

(24)

Note this analysis does not consider cycle slips, which would make the value of \( M \) probabilistic. Instead, we will simply assume that measurements are monitored and excluded if a cycle slip appears to have been likely. Given that continuous carrier tracking is not required, such an exclusion should have minimal impact on algorithm performance.

Although the propagation of the integer estimate \( \delta \hat{L} \) is relatively straightforward, according to (24), relating the integer estimates \( \delta \hat{L} \) back to the state estimates \( \hat{x} \) requires additional effort. To use (24), it is first necessary to compute the probability associated with each integer combination \( \delta \hat{L} \), since many grid points share the same integer given a sufficiently resolved histogram grid. The total probability associated with each integer combination \( \delta \hat{L} \) is

\[
p_{+}(\delta \hat{L}_{k-1}) = \iint_{\Omega(\delta \hat{L}_{k-1})} p(\delta \hat{x}_{k-1})d\delta V
\]

(25)

where

\[
\Omega(\delta \hat{L}_k) = \left\{ \delta \hat{x}_k \mid \text{floor} \left( \frac{(G\delta \hat{x}_k + \hat{e}_{\phi,k})/\lambda}{\lambda} \right) = \delta \hat{L}_k \right\}
\]

(26)
and where the error estimate $\hat{e}_{\phi,k}$ is computed using (20).

The prior distribution for the state estimate $\hat{x}_k$ at each new epoch $k$ is computed by distributing the probability from (25) evenly across the volume associated with the updated integer.

$$p(\hat{\delta x}_k) = \frac{p_x(\delta \hat{L}(\delta \hat{x}_k) - \delta M_k)}{\int_{\Omega(\delta \hat{x}_k)} dV}$$

(27)

The denominator describes the volume of the state space associated with each integer, over which the probability form (25) is uniformly distributed. In general, this volume is different for different integers (even in the limit where the integral is evaluated on a continuous space rather than a discretized space).

The complete Bayesian algorithm can thus be implemented by sequentially applying the prediction equations (24), (25) and (27), to update the old state-estimate distribution to the current epoch and then by applying the measurement update step of (14) to integrate new sensor data. In effect, the propagation step imposes a nonlinear integer constraint that was not captured in the least-squares estimate of the float solution (12).

SIMULATION-BASED COMPARISON

Simulations were used to analyze and compare the performance of the proposed Bayesian methodology to that of the conventional float solution. The goal of the analysis is to evaluate the rate at which the accuracy and error bounds for each method converge.

Simulation Setup

In the simulation, eight GPS satellites were simulated to be visible above the horizon. All these satellites were assumed to remain visible across the entire period of the simulation. For this baseline study, continuous carrier-phase tracking was assumed, so that code and carrier-phase measurements were available at all epochs. Simulated GPS measurements were
subject to randomly generated Gaussian measurement errors. The differential code-phase measurement error was assumed to be 1.0 m (one sigma) for all satellites. The differential carrier-phase error was assumed to be 2 cm (one sigma).

In order to limit the size of the PDF volume, two additional pieces of navigation data were introduced. First, it was assumed that the user receiver moved parallel to a ground plane at a known height, subject only to small random vertical perturbations with a standard deviation of 1 cm. Second, a vision-based sensing system capable of detecting reliable landmarks (i.e. road lane markers) was assumed to be collocated with the GPS receiver, as in [13,14]. This system was assumed to provide cross-track measurements with an error of 0.20 m (one sigma). These additional measurements were introduced simply to compress the volume of the histogram PDF model, by a factor of approximately two in the horizontal direction and by nearly two orders of magnitude in the vertical direction.

All sensors were assumed available to both algorithms, one which computed the conventional float solution and the other which computed the Bayesian solution.

For the purposes of computing the Bayesian state-estimate PDF, a histogram model with $10^9$ voxels was used. Voxel coordinates were spaced with uniform separation (2 cm) in all four state-estimate dimensions (three spatial dimensions and time). The bounding dimensions of the volume were set to $\pm 7$ m for the assumed direction of travel (along-track coordinate) and for the clock bias ($b$ coordinate). The bounding dimension in the lateral direction (cross-track coordinate) was set to $\pm 3$ m; that for the vertical direction (up coordinate), to only $\pm 10$ cm.

**Simulation Results**

As might be expected, the Bayesian histogram filter required vastly more computational resources than the least-squares method. Data from each epoch was processed in a fraction of a second when computing the float solution with a conventional laptop; by comparison, each epoch was processed in approximately 8 minutes when computing the Bayesian solution using the same laptop.

In exchange for burdensome processing, the Bayesian algorithm provided great performance benefits, achieving a much higher accuracy after only a few epochs of processing. Fig. 1 provides a graphical comparison of the PDFs generated by both algorithms in processing data from four epochs. The epochs were assumed to be separated over a sufficiently long interval such that all measurement errors are effectively independent. PDFs for the float solution (left column) and Bayesian approach (right column) are illustrated for each of four epochs (corresponding to the four rows of plots). The figure shows
PDFs projected into the two-dimensional horizontal plane (along-track and cross-track directions). The illustrated PDF is a marginal probability on the horizontal plane, with probabilities integrated across the remaining two state dimensions (up coordinate and clock bias). In the illustration, dark regions indicate high probability densities, and light regions indicate low densities. Grayscales are graded logarithmically, to enhance the visibility of the PDFs across a wide range of scales. In the simulation, GPS receiver location is allowed to change from one time step to the next; the true location of the GPS receiver in each plot is identified by a circle marker.

![Projection of Float PDF](image1.png) ![Projection of Bayesian PDF](image2.png)

**Fig. 1. Comparison of float solution (left) to Bayesian solution (right) for four sequential time steps**

PDF appearance gives a qualitative indication of the differences between the float solution and the Bayesian solution. The most salient difference between the two algorithms is that the Bayesian solution is speckled (multi-peaked), whereas the float solution estimates position as a single-peaked Gaussian distribution. Each peak in the Bayesian distribution corresponds to a different integer combination.
At the first time step, both algorithms exhibit PDFs with approximately elliptical contours of constant probability. Elliptical contours are expected for the float solution, which inherently models the state-estimate PDF as Gaussian distributed. Though the Bayesian PDF need not be Gaussian distributed in general, it has this initial appearance because the dominant error source at this step is the Gaussian-distributed code measurement error.

The distinction between the float and Bayesian solutions becomes more apparent in the second and subsequent time steps. Bayesian state propagation occurs for the first time between the first and second time steps. This propagation is critical to contracting the Bayesian PDF, because it is the propagation step that enforces the integer constraint.

In effect, the propagation step performs a form of integer ambiguity resolution. This ambiguity resolution appears as a reduction in the number of likely integer combinations (dark peaks) visible in Bayesian PDF in the second and subsequent epochs. Only a handful of peaks are visible at the fourth time step. By the fourth time step, it is clear that the Bayesian solution is converging to the correct integer, as can be identified in Fig. 1 by noting that the highest density (darkest) region of the PDF coincides with the true position (circle marker).

To quantify accuracy more precisely, it is useful to plot Cumulative Distribution Functions (CDFs) for the two approaches. CDF comparisons are presented in Fig. 2 and Fig. 3. In these figures, the CDF is evaluated by integrating the planar PDF from Fig. 1 outside an elliptical region of scalable area, for which the elliptical contour is defined by a Mahalanobis distance \( m \).

\[
CDF = \int_M^\infty p(m)dm
\]  

\[m = \left( (x - \mu)^T \Sigma^{-1} (x - \mu) \right)^{\frac{1}{2}}\]  

For the float solution, the Mahalanobis distance is computed with \( x \) being the two-dimensional horizontal state and with \( \Sigma \) and \( \mu \) being the covariance and mean estimate of that solution at a particular epoch. (Each value of \( m \) thus corresponds to an
elliptical contour of constant probability.) For the Bayesian solution, the Mahalanobis distance is evaluated using the same parameters at each epoch, except that the mean $\mu$ is shifted to the location of the mean of the Bayesian histogram.

At the first epoch, the CDF curve is very similar for both methods, as shown in Fig. 2. The figure depicts both CDF curves using Mahalonobis coordinates. In these coordinates, all Gaussian distributions are identical, due to normalization by mean and covariance according to (29). The float solution produces a Gaussian CDF, which tracks this characteristic curve. The Bayesian solution roughly tracks the same characteristic curve; however, the CDF is nonsmooth, with steps visible corresponding to local PDF peaks and valleys (associated with particular integer combinations).

Fig. 2. Comparison of Float and Bayesian CDFs at First Epoch

The CDF curves for the float and Bayesian solutions diverge after the first epoch (see Fig. 3). The float solution consistently produces a Gaussian curve, and hence the Mahalanobis CDF remains in the same location, even though the PDF mean and covariance evolve over time. By contrast, the Bayesian solution increasingly shifts weight to particular integer combinations, such that the distribution becomes increasingly non-Gaussian.
A salient feature of Fig. 3 is that the Bayesian CDF asymptotes at a minimum probability for large values of the Mahalanobis-normalized error. This behavior is a result of the numerical implementation of the Bayesian algorithm, which clusters low-likelihood peaks together into an “unassigned” probability with the purpose of enhancing computational efficiency. Because the unassigned probability is not associated with a particular integer, it is excluded from the CDF (such that the CDF always provides a conservative bound on error).

![CDFs for Subsequent Time Steps](image)

**Fig. 3. Comparison of Float and Bayesian CDFs at Epochs 2 through 5**

**Discussion**

A significant result is that the accuracy of the Bayesian solution improves much more rapidly than that of the float solution. This phenomenon is evident in Fig. 3, which compares the Mahalanobis CDFs for epochs 2 through 5. For each successive epoch, the CDFs are normalized by the same (improving) covariance matrix. Despite the normalization, the Bayesian solution still compresses toward the left side of the figure, indicating improved accuracy (i.e., lower probabilities of large errors) relative to the float solution. This result has important implications for the speed of integer ambiguity
resolution, as the Bayesian algorithm may greatly reduce the number of epochs needed to achieve high-precision GNSS positioning as compared to conventional integer ambiguity resolution techniques.

The analysis shown in Fig. 3 is also promising for high-integrity applications. As illustrated in the figure, the Bayesian solution has very tight error bounds even in the far tails. Consider the error bound that contains all except a small risk probability of $10^{-7}$. At the level of $10^{-7}$ probability on the vertical axis of Fig. 3, the Bayesian solution has an error that is an order of magnitude smaller than the corresponding float solution error at the same risk probability.

**SUMMARY**

This article identified a new class of algorithms dubbed simultaneous position estimation and ambiguity resolution (SPEAR) algorithms. As an example of a SPEAR algorithm, the paper introduced a novel Bayesian filtering technique designed to estimate user receiver position subject to integer propagation constraints. In this algorithm, integer estimates are deterministically related to receiver-state estimates. The receiver-state estimate is itself a distribution, represented using a nonparametric histogram model in order to capture the distribution’s distinctly non-Gaussian character.

The Bayesian positioning algorithm offers two substantial benefits. First, because the algorithm represents integers as a function of receiver states, the algorithm does not “fix” integers, and hence it is robust to events that interrupt carrier-phase tracking for one or several satellites. Second, because the algorithm estimates position subject to an integer constraint, it more fully exploits available measurement data, allowing for convergence to high accuracy in many fewer epochs than would be possible using traditional algorithms. The major liability of the proposed Bayesian estimator is its very large computational burden.

**ACKNOWLEDGEMENTS**

The author gratefully acknowledges NASA (Contract 11-1-A3.01-8393 ARC) for supporting this research. The opinions discussed here are those of the authors and do not necessarily represent those of the FAA or other affiliated agencies.
REFERENCES


