Modal/Eigenstate Determination for Reoccurring Dynamics

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Executive Summary- Nearly by definition, soft robots are under-actuated. That is, they have significantly fewer actuators than degrees of freedom. One way to control un-actuated states in soft robots is to design the dynamics of the physical robot system (structure and materials) to complement available actuators. In this paper we introduce a modeling approach for this class of design problem, which we call Modal/Eigenstate Determination for Reoccurring Dynamics (MEDFRD).

Keywords- Co-Design, Embodiment, Eigenstructure Assignment, Under-actuation, Soft Robot

I. Introduction

In broad terms, a soft robot is any robotic system that is capable of a gross deformation of its structure. This definition allows for highly jointed semi-rigid systems as well as continuum systems with infinite degrees of freedom [1]. Such highly deformable structures promise practical benefits, such as the ability in Urban Search and Rescue applications to squeeze through gaps in rubble smaller than the robot’s nominal cross-section [2, 3].

Currently, methods are needed to enable effective locomotion of soft robots through complex terrain and to do so with as few actuators as possible, given strict size, weight, and power constraints [4-6]. Controlling a highly deformable soft robot with a minimal number of actuators
is challenging, because such robots feature many more degrees of freedom than actuators. In other words, such systems are inherently under-actuated.

This paper introduces a conceptual approach for designing under-actuated dynamic systems, such as soft robots, in order to enable those systems to follow desired motion trajectories, trajectories that could not otherwise be tracked using traditional feedback control methods [7, 8], alone. Our particular approach is to specify the desired motion pattern as a resonance mode resulting from the system’s combined passive (structural) and active (feedback) dynamics. For a linear dynamic system, the desired motion pattern is thus specified as a set of Eigenstates, each comprised of an Eigenvalue-Eigenvector pair [9]. For nonlinear dynamic systems, motion patterns might also be specified in terms of a modal analysis; however, it is important to recognize that modal properties (frequency, phasing etc.) are generally a function of amplitude for nonlinear systems [10]. An advantage of using mode-based movement patterns is that they have been demonstrated to be resilient, inherently rejecting certain impulsive disturbances. Previous efforts to apply mode-based analyses to realize desired movement patterns have focused exclusively on fully actuated systems [11, 12]; such tools do not yet exist to achieve similar results in under-actuated systems.

This paper seeks to extend modal-control concepts to under-actuated dynamic systems by leveraging concepts from the emerging discipline of co-design. Co-design merges passive structural design, materials selection, and feedback control design [13, 14]. The term co-design has its origin in the mechanical design and feedback controls communities; the similar term embodiment is sometimes used within the robotics community to describe machine
“intelligence” derived from physical structure [15]. Co-design techniques have recently been developed to reduce energy inputs and to create minimal changes to physical structures that enable systems to accomplish otherwise impossible tasks [16, 17].

Co-design methods have not yet been applied either to modal tuning or to under-actuated system design. Thus to achieve our goal, we use co-design to unify three additional research areas, as shown in Figure 1: passive modal design, active Eigenstructure assignment, and feedback design for under-actuated systems.

Along the spectrum of under-actuated systems, the extreme cases are those with no actuation at all (passive dynamic systems) and those with full actuation. Fortunately, our proposed concept can benefit from prior research conducted at each end of the spectrum, including both the fully passive and fully actuated extremes. For instance, tools have been developed for passive dynamic systems that adjust local mass or stiffness to alter mode shapes, resonant frequencies, or
decay rates [18], either to attenuate resonance (where it is undesired, as in a bridge) or to amplify resonance (where it is desired, as in energy harvesting) [19]. Similarly, prior research has developed several tools to assign Eigenstates using active control in fully actuated systems [20-26]. Our approach bridges these fully passive and fully actuated extremes, combining the best features of both sets of approaches. As a result, our method greatly expands the design space as compared to existing methods for controlling under-actuated systems, which typically rely on partial feedback linearization and Eigenvalue (but not Eigenvector) placement [27, 28].

The remainder of this paper details our approach for applying modal analyses to embed desired reference trajectories into the combined passive and active dynamics of an under-actuated system. In a first section, a general approach is described. The general approach is intended to apply widely to diverse classes of dynamic systems. Subsequently, for purposes of demonstration, the approach is developed in detail for a simple class of model system (the linear time-invariant, or LTI, system). As a representative example, this approach is applied to motion pattern determination for a multi-link pendulum system.

III. MEDFRD Problem

In this section we introduce a mathematical representation that describes the problem of designing an under-actuated system to follow a desired pattern of internal motion (e.g., a desired gait). Our approach uses modal analysis as an intermediate step. We refer to the resulting quantitative design problem as Modal/Eigenstate Determination for Reoccurring Dynamics (MEDFRD).
1. General Statement of MEDFRD Problem

This section describes the characteristics of the MEDFRD problem for a general under-actuated system. Here it is assumed that motion results from periodic forcing that excites resonance. Note that in the general case, the system dynamics are not restricted; they may be modeled by any differential equations that exhibit resonance (linear or nonlinear, continuous-time or discrete-time, partial or ordinary).

The primary goal of the MEDFRD problem is to match resonant motion patterns to motion patterns that have been specified by a designer. To achieve this goal, five inputs to the design problem need to be identified, as enumerated below. The primary two inputs are the following.

1. **Periodic motion specifications** $\Omega_{des}$. It is assumed that the desired reference trajectory can be mapped into a modal structure $\Omega_{des}$, which describes the desired natural motions of the dynamic system.

2. **List a of tunable parameters.** Parameters that can be modified through design of structure, materials, and feedback control laws must be identified.

The tunable parameters determine the actual modal structure of the system $\Omega_{act}(a)$, which is to be matched to the desired modal structure $\Omega_{des}$. A metric is needed to identify the quality of the match.
3. **Metric $J$ identifying quality of match.** The metric $J$ must be defined to compare how closely $\Omega_{act}(a)$ matches $\Omega_{des}$.

It is further important to account for limits that may exist on the values of the vector $a$ of tunable parameters. These limits fall into two categories.

4. **Constraints on parameter values.** Parameter values may be subject to equality constraints $g(a)$, such as those imposed by kinematic relationships, and also to inequality constraints $f(a)$, such as those imposed by practical design considerations (e.g. mass must be positive, lengths may be bounded, etc.)

5. **Attenuation for undesired dynamic modes.** Specifications may determine only a subset of system modes, so the energy $V_i$ associated with each unspecified mode $i$ should quickly decrease, to ensure these modes decay quickly away.

Combining these elements, MEDFRD can thus be framed as an optimization problem to which a numerical solution can be sought.

$$\min J(\Omega_{des}, \Omega_{act}(a))$$
$$f(a) < 0$$
$$g(a) = 0$$
$$\frac{\partial V_i}{\partial t} < 0 \ \forall i$$

(1)
2. MEDFRD Problem for LTI Systems

Obtaining a practical solution to the MEDFRD problem requires a mathematical model of system dynamics. As a starting point, it is logical to first consider the MEDFRD problem for the simplest class of dynamic systems: linear time-invariant systems (LTI systems).

The following equations model the dynamics of an LTI system. Here the state vector \( \mathbf{z} \) is governed by the set of ordinary differential equations

\[
\dot{\mathbf{z}} = A\mathbf{z} + B\mathbf{u},
\]

(2)

where \( A \) and \( B \) are time-invariant matrices and where \( \mathbf{u} \) is a vector of actuator input signals. With the intention of preserving the LTI model structure, we assume the input to be full-state, linear feedback superimposed on a sinusoidal excitation signal \( \omega \).

\[
\mathbf{u} = -K\mathbf{z} + L\omega.
\]

(3)

Here \( K \) is a constant, real gain matrix and \( L \) is a constant, complex matrix that shifts the amplitude and period of \( \omega \). The sinusoidal excitation \( \omega \) is assumed to be a set of \( N \) harmonics, where the periods of all excitations are integer multiples of a base period \( \omega_0 \).

\[
\omega = \begin{bmatrix}
\mathbf{e}^{\omega_0 t f} \\
\mathbf{e}^{2\omega_0 t f} \\
\vdots \\
\mathbf{e}^{N\omega_0 t f}
\end{bmatrix}.
\]

(4)
The Eigenstates of the closed-loop system are thus the Eigenstates of a matrix $\tilde{A}$:

$$\tilde{A} = A - BK.$$  \hspace{1cm} (5)

Achieving the desired motions is thus a matter of solving the co-design problem to obtain the parameters in $\tilde{A}$ that closely approximate the desired motion patterns when the system’s modal dynamics are excited by $\omega$. As described in the previous section, the two primary inputs to this design problem are (1) the desired modal structure $\Omega_{des}$ and (2) the list of tunable parameters $a$.

To simplify notation, the list of tunable parameters is assumed to include all elements of the matrix $\tilde{A}$; any non-tunable parameters within this matrix are simply fixed to a constant value by an equality constraint in $g(a)$. The modal structure $\Omega_{des}$ is specified in terms of a desired Eigenvector matrix $\tilde{X}$ and a desired Eigenvalue matrix $\tilde{\Lambda}$. In other words, if an exact match is achieved,

$$\tilde{A}\tilde{X} = \tilde{X}\tilde{\Lambda}. \hspace{1cm} (6)$$

In addition to $\tilde{X}$ and $\tilde{\Lambda}$, the set of modal properties also includes $L$, the matrix which sets the amplitudes and phases of the excitation signal $\omega$.

$$\Omega_{des} = \{\tilde{X}, \tilde{\Lambda}, L\} \hspace{1cm} (7)$$

These modal parameters can be directly related to the state trajectories using linear systems theory, as described in [29].
Given that an exact match of the desired and actual dynamics may not be possible, the designer must introduce a metric $J$ to evaluate the quality of the approximation. For instance, a Frobenius norm, indicated below by the subscript $F$, might be used to compute the following cost function as a relaxation of equation (6).

$$J = ||\bar{A}X - \bar{X}\bar{A}||_F$$

(8)

The Frobenius matrix norm is computed as the root-sum-of-squares of the elements of a matrix. Note here that the cost function does not consider the matrix $L$, as it is assumed that the phases and amplitudes of the excitation signal can be set arbitrarily by the system designer, such that an exact match to the desired $L$ is possible.

The cost function steers the selection of the parameters in the $\bar{A}$ matrix to match the desired modal parameters; however, the cost function does not respect inherent constraints, which are functions of the structure, materials and feedback-control design of the system. These practical considerations are modeled simply as inequality and equality constraints on $\bar{A}$.

$$f(\bar{A}) < 0$$
$$g(\bar{A}) = 0$$

(9)

A more difficult limitation, from a numerical perspective, is to ensure that unspecified dynamics are mitigated. For an under-actuated system, it is not generally possible to define the entire modal structure of the dynamic system (2). In such cases, it may be desirable to define $\bar{X}$ and $\bar{A}$
to specify only a partial system of Eigenvectors and Eigenvalues. Note that equations (6) and (8) are purposefully expressed in a form that is valid even if an incomplete set of Eigenstates is specified (i.e. even if $\mathbf{X}$ and $\mathbf{\Lambda}$ are not full rank).

Although certain Eigenstates may not be explicitly specified, it is nonetheless important that their associated energy quickly decays. In the general MEDFRD problem, this decay was described by a Lyapunov energy function $V_i$. For LTI systems, this constraint on the decay of modal energy is equivalent to the stability condition that requires all system Eigenvalues to have negative real parts. Mere stability is insufficient, however. In order to achieve desired motions in a reasonable time, it is necessary that the unspecified modes be not only stable, but that they decay on a rapid timescale. Defining this timescale to be $\phi^{-1}$, a settling-time constraint can be imposed with the following form:

$$\max_{k \in K} (\text{real}(\lambda_k)) < \phi,$$

(10)

Here the set $K$ includes all unspecified Eigenstates.

Thus, customizing the five MEDFRD inputs to applications involving LTI systems results in the following optimization problem.
\[ \min J \]

\[ J = \| \tilde{A} \tilde{X} - \tilde{X} \tilde{A} \|_F \]

s.t \[ \max_{k \in K} \text{real}(\lambda_k) < \phi \]

\[ f(\tilde{A}) < 0 \]

\[ g(\tilde{A}) = 0 \]

In general, this problem is nonconvex and must be solved with an appropriate global search or global optimization algorithm. In the specific case in which all system Eigenvalues can be set (\( \tilde{A} \) is full rank) and in which inequality constraints are non-essential, then a direct solution approach called Dual Domain Eigenstate Factorization [30] can alternatively be applied to solve (11).

V. Model System

Although a more sophisticated (nonlinear, continuum) model is needed to capture the detailed dynamics of a soft robotic system, intuition can be obtained by applying MEDFRD to a multi-link pendulum system, which can be viewed as an approximately linear, highly simplified model for a crawling robot.

This section details the application of MEDFRD to a three-link pendulum. This model system has several tunable parameters, which include the length, mass, and inertia of each link, as well as a spring constant modeling the flexibility of the joints between each link, and joint torques between each link. An illustration of the three-link pendulum system can be seen in Figure 2, with joint torques labeled as \( T_i \).
To obtain a wave like progression of moving linkages, an oscillating mode shape for the three-link pendulum was selected. This oscillating mode shape was defined by an Eigenvalue, which determined the frequency of oscillation of the three links, and by an Eigenvector, which determined the amplitude of the oscillation of each link as well as the relative phasing of each link’s swing. The Eigenvalue selected for this case was lightly damped, with a motion repetition frequency of $\pi/2$ rad/sec (i.e., specified eigenvalue was $\lambda = -0.1 + \frac{\pi}{2}j$). The Eigenvector selected was

\[
x = \begin{bmatrix}-.10 + 1.57j \\ -1.18 + 1.04j \\ -1.57 - .10j \\ 1.00 \\ .70 + .70j \\ 1.00j\end{bmatrix}.
\]  

(12)
Both equality and inequality constraints were imposed on $\tilde{A} \in \mathbb{R}^{6 \times 6}$. Half of the rows of $\tilde{A}$ are kinematic equations relating angle to angular speed; the other half of the rows are moment balances. The kinematic equations cannot be influenced through tuning of physical or feedback parameters. Elements of these rows are one or zero, values imposed as equality constraints in $g(\tilde{A})$. Inequality constraints on physical and control parameters were also considered. The ranges of parameter values allowed by inequality constraints are summarized in Table I.

<table>
<thead>
<tr>
<th></th>
<th>Mass</th>
<th>Length</th>
<th>Inertia</th>
<th>Spring Constant</th>
<th>Control Gains</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$[0.01, \infty]$</td>
<td>$[0.01, \infty]$</td>
<td>$[0.01, \infty]$</td>
<td>$[0.01, \infty]$</td>
<td>$[-0.1, \infty]$</td>
</tr>
</tbody>
</table>

The additional constraint (10) was introduced to attenuate unspecified Eigenstates. Specifically, rapid decay of unspecified modes was ensured by requiring all unspecified Eigenvalues to have a real part below $\phi = -0.4$.

VI. Results

For the three-link pendulum system, an exact match to the desired Eigenstates is not generally possible unless all joints are fully actuated. In this case study, however, an under-actuated system was considered, with only the first two joints (starting at the base) actuated. The quality of the MEDFRD solution for this system can be assessed by comparing the optimized system Eigenstates to the target Eigenstates. Figure 3 illustrates the specified Eigenstates (red stars) next to the computed Eigenstates (blue circles). The two specified Eigenvalues are well
approximated; the remaining four eigenvalues all satisfy the $\phi$ constraint (shown as a green vertical line).

Figure 3B illustrates the Eigenvectors, with relative amplitude (normalized by amplitude of first state) illustrated on the vertical axis and phase difference (compared to first state) on the horizontal. As can be seen in the figure, the second and third components of the computed Eigenvector match phase within 0.1 rad for link 2 and 0.5 rad for link 3. The relative amplitude for link 2 is off by 12% while the relative amplitude for link 3 is off by 91%.

Figure 3: Representative Solution solving MEDFRD problem for linearized 3-link pendulum
VII. Discussion

In the example considered, the under-actuated system was tuned so that its dynamics were a relatively close match for those specified. Eigenvalue parameters and relative phasing were matched particularly well; relative amplitude was matched less well. In this case the quality of the amplitude match was not significantly affected by the inequality constraints of Table I; however, it is clear from Figure 3A that one of the modes has an Eigenvalue that “bumps up” against the attenuation constraint ($\phi \leq 0.4$).

The key qualitative characteristic is that the modal properties of the solution recreate a traveling wave motion pattern, with phasing progressing in the desired direction from the pendulum anchor to its tip. Such traveling wave patterns are characteristic of the gaits of soft, crawling animals [31], and might be leveraged for design of soft biomimetic robots in the future.

Further work is needed to extend the methods presented for application to soft robot systems. Although the detailed approach presented here is tailored to linear systems, the more general MEDFRD problem, as specified by (1), could be adapted to model nonlinearities (such as contact with the environment) and continuum systems with infinite degrees of freedom. A key benefit is that the use of a modal representation of system dynamics reduces the dimensionality of the system to match the number of actuators available; in other words, the number of system dynamic degrees of freedom is less important than the number of actuators in creating a desired reference trajectory. Detailed development of the MEDFRD problem for a wider range of dynamic systems is left as future work.
VIII. Conclusion

In this paper the problem of generating desired periodic motion for an under-actuated system is addressed by a generalized co-design method, which defines the system’s modal structure through tuning of feedback and physical parameters. The approach generates a mathematical representation of the design problem, which we call Modal/Eigenstate Determination for Reoccurring Dynamics (MEDFRD). The MEDFRD problem is characterized for general dynamic systems and developed in more detail for LTI systems. A three-link pendulum case study demonstrates that the MEDFRD problem can be solved to obtain useful motion patterns, such as traveling wave gaits.

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References


