Defining Shapeability in Eigenstate Specification for Linear Systems

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ABSTRACT
This paper introduces a metric called degree of shapeability to determine whether or not it is possible to assign a set of desired eigenstates to a particular linear dynamic system. Specifying an eigenstate, which consists of an eigenvalue and its associated eigenvector, can be essential in designing certain vibrating structures and undulating mechanisms. In general it may be desirable to specify more than one eigenstate. The number of eigenstates that can be specified for a particular linear system is not always immediately evident, however, because of physical and kinematic constraints, as well as actuator placement. To quantify the potential for active feedback and/or passive parameter tuning to influence a system’s eigenstates, we identify a metric, the degree of shapeability, that relates the structure of the linear system to the number of eigenstates that a designer may fully define. We investigate the effect of system equality and inequality constraints on the ability to specify eigenstates for linear dynamic systems.

I. INTRODUCTION
A system’s eigenstates, comprised of both eigenvectors and eigenvalues, are the basis for repeated movement patterns [1]. Eigenstate-based movement patterns are dynamic, inherently rejecting certain environmental disturbances, and hence are appropriate for applications such as the locomotion of robotic platforms in rough terrain. For many robotic platforms it is difficult to generate dynamic, repeated movement patterns because of the physical constraints associated with design. This is especially true for underactuated systems in which the number of degrees of freedom exceeds the number of actuators. For these cases, no method currently exists that allows for the specification of eigenstates.

Caterpillar-inspired robots are one example of a robotic system that locomotes using a repeated motion pattern [2]. A pneumatically actuated caterpillar-like robot, constructed with silicon elastomer, is illustrated in Figure 1. Such a robot has potential to squeeze through small spaces and traverse complex terrain. Caterpillar robots experience certain nonlinear forces (e.g. ground contact forces); however, we hypothesize that linearized models of such systems may be sufficient in designing robust, dynamic gaits. As such, this paper will focus on linearized dynamic systems and specifically on the impact of design constraints and underactuation on eigenstate assignment.

Figure 1: Pneumatically Actuated Deformable Robot

Eigenstates for a system can be altered by either tuning passive physical parameters or by incorporating active feedback control. Tuning passive physical parameters of a system, such as element masses or stiffnesses, can allow alteration of eigenmode shape, resonant frequency, and decay rate [3]. When altering physical parameters, constraints often limit feasible solutions. These constraints are generally the result of kinematic coupling, size requirements, or fundamental physics (e.g. mass must be positive). In order to
explore a wider design space given these constraints, it is desirable to incorporate feedback control.

Introducing feedback control can effectively modify passive parameters to assist in achieving a desired set of eigenstates. The ability of feedback control laws to place eigenvalues is well understood [4-7]. Control over eigenvectors (in addition to eigenvalues) is a relatively new concept, referred to as modal control [8-11]. Conventional modal control design methods focus on altering the control gains for full-state feedback of linear systems, but neglect the tuning of physical parameters. To merge the benefits of passive-parameter tuning and modal control, we have recently introduced a method for simultaneously tuning both physical and feedback parameters in designing specified eigenstates into a physical system. Our method is called Modal Eigenstate Determination for Recurring Dynamics (MEDFRD) [12].

Underactuation is the major challenge in using feedback control to specify eigenstates. As an example of an underactuated system, again consider the pneumatic robot from Figure 1. The robot system is inherently underactuated since the soft elastomer structure has, effectively, an infinite number of degrees of freedom. The system becomes more severely underactuated if selected pneumatic pistons are removed. Eventually, if all of the actuators are removed from the robot, the robot’s eigenstates are purely a function of its physical parameters.

This paper addresses an important gap in the existing literature. Our approach is to characterize linear systems by quantifying system shapeability, which refers to the number of eigenstates which can be assigned given design constraints and underactuation. The remainder of this paper will develop the concept of degree of shapeability and explore the differing impacts of equality and inequality constraints on eigenstate assignment. First, we will briefly overview the MEDFRD method. In Section III we will introduce degree of shapeability. Section IV will apply this concept to analysis of a model three-link pendulum system, and Section V will discuss the results. The paper will conclude with a brief discussion of the capabilities and limitations of the proposed methodologies.

### II. MEDFRD METHOD

This section provides a brief review of the MEDFRD method, originally proposed in [12]. The MEDFRD method shapes the dynamics of a state-space system, which are written as:

\[
\dot{z} = Az + Bu
\]

where \(z\) is the state vector and \(u\) is the vector of control inputs. If we assume full-state feedback control \((u = -Kz)\) the dynamic system can be simplified as follows.

\[
\begin{align*}
\dot{z} &= \tilde{A}z \\
\tilde{A} &= A - BK
\end{align*}
\]

For a fully actuated system in which all parameters of the matrix \(\tilde{A}\) can be tuned without constraints, the problem of specifying a desired set of eigenstates is one of satisfying the following equation. The number of specified eigenstates \(Q\) may be equal to or smaller than the dimension \(D\) of the \(\tilde{A}\) matrix.

\[
\tilde{AX} = \tilde{X}\tilde{A}
\]  

Where \(\tilde{X}\) and \(\tilde{A}\) represent rectangular matrices of specified eigenvectors and eigenvalues. Equation 3 has the property that multiplying any column by an arbitrary scalar has no effect, because both sides of (3) are multiplied in the same manner.

The problem is made more complex by physical system constraints (e.g. positive masses and kinematic constraints). To account for constraints the exact equation (3) can be replaced with an optimization problem. This optimization problem has the following form, where \(J\) is the cost function, which depends on the variable elements of the \(\tilde{A}\) matrix and the specified (fixed) eigenstate parameters of the \(\tilde{X}\) and \(\tilde{A}\) matrices.

\[
\begin{align*}
\min_{\tilde{A}} J \\
f(\tilde{A}) &< 0 \\
g(\tilde{A}) &= 0
\end{align*}
\]

In this formulation, all inequality constraints are lumped into the vector \(f\), and all equality constraints are lumped into the vector \(g\). Constraints in \(f\) are generated when constants are present in the \(\tilde{A}\) matrix. For example, if element \(\tilde{A}(1,3) = 0\) then one of the rows of \(g(\tilde{A})\) is the linear equality \(\tilde{A}(1,3) = 0\). Constraints in \(g\) are generated when control gains or physical parameters are given upper or lower feasible ranges, thus limiting achievable values for elements of the \(\tilde{A}\) matrix. For example, if \(\tilde{A}(2,3)\) is a mass (which must be positive), then one of the rows of \(f(\tilde{A})\) is the linear inequality \(\tilde{A}(2,3) > 0\).

The solution to this optimization problem depends on the specific choice of the cost function, \(J\). Various cost functions might reasonably be considered. In this paper, we select a representative case: a cost function based on the Frobenius norm of the difference between the two sides of equation (3), which in effect compares the actual system dynamics to the desired eigenstates.

\[
J(\tilde{A}; \tilde{X}, \tilde{A}) = \|\tilde{AX} - \tilde{X}\tilde{A}\|_F
\]

The Frobenius matrix norm is defined as the root-sum-of-squares of the elements of a matrix.

The optimization problem (4) must be further refined to account for two additional details. First, the solution to (4) is not unique, because additional degrees of freedom typically remain even after the optimization is satisfied. Second, the optimization only considers specified dynamics, and hence it is possible that the remaining unspecified dynamics might dominate the specified dynamics. This would be true, for
example, if unspecified eigenvalues were neutrally stable (near zero) or unstable (positive). For this reason, it is desirable to make the real part of the unspecified eigenvalues large and negative, thus ensuring the unspecified dynamics decay away as quickly as possible. In this regard, additional degrees of freedom in the optimization process are an asset, as they make it possible to minimize the unspecified eigenvalues.

A multi-objective optimization problem is the result of desiring both to specify eigenstates as closely as possible and also to mitigate unspecified dynamics. A first optimization criterion minimizes the discrepancy between the system’s actual eigenstates and the target eigenstates specified by the designer. A second optimization criterion maximizes the decay rate of all unspecified eigenstates by minimizing their eigenvalues. A reasonable choice of these cost functions, $J_1$ and $J_2$, can be seen in equation (6).

$$J_1 = \| \bar{\mathbf{A}} \bar{\mathbf{X}} - \mathbf{X} \bar{\mathbf{A}} \|$$

$$J_2 = \max_{i,k} \left( \text{Real}(\lambda_i) \right)$$

(6)

In theory, the solution that describes the minimum cost of the first optimization criterion ($J_1$), for each possible minimum value for the other optimization criterion ($J_2$), is a Pareto optimal curve. However, in this paper, due to the nonconvexity of $J_2$, only an approximation of the Pareto optimal curve was found. An approximate Pareto optimal curve can be generated by solving the following optimization problem for a range of values of the parameter $\phi$, which represents the maximum allowable real component of the unspecified eigenvalues.

$$\min J_1$$

$$J_2 < \phi$$

$$f(\bar{\mathbf{A}}) < 0$$

$$g(\bar{\mathbf{A}}) = 0$$

(7)

We refer to this control design problem and our solution method as Modal Eigenstate Determination for Recurring Dynamics (MEDFRD). The MEDFRD method requires computation of a solution to equation (7) to specify desired system eigenstates while mitigating unspecified dynamics. A full description of the method can be found [12].

III. EIGENSTATE SPECIFICATION

A. Degree of Shapeability

When applying the MEDFRD method it is useful to the designer to know how many eigenstates can be specified. To answer this question in a formal, quantitative manner, we define a degree of shapeability metric. This metric might be used for example to determine the number of actuators required to achieve a particular undulating gait for the caterpillar-like robot seen in Figure 1.

Definition 1: A system’s degree of shapeability is the number of eigenstates that can be fully specified.

Here fully specified means that all eigenstate parameters (the eigenvalue and all elements of the eigenvector) can be set independently of one another and of all other system eigenstates.

The degree of shapeability is affected by equality and inequality constraints acting on the system. In general both are important in solving optimization problem (7). Equality constraints strictly limit the number and location of free parameters in the $\bar{\mathbf{A}}$ matrix, and hence determine the maximum number of assignable eigenstates. Inequality constraints do not necessarily limit the number of assignable eigenstates unless an inequality constraint becomes active. To understand the role of inequality constraints on the shapeability of a system, an example is helpful. Suppose a system has no active inequality constraints for a selected eigenstate; then the degree of shapeability is simply equal to that determined by the equality constraints. On the other hand suppose all inequality constraints become active. There may be no feasible way to attain any independent eigenstate in this situation and the degree of shapeability may drop to zero. Effectively, the number of active inequality constraints and the location of those constraints in the $\bar{\mathbf{A}}$ matrix creates a range of shapeability, varying from the shapeability determined by equality constraints alone, to the shapeability achievable when all physical and control constraints are held constant (generally zero).

Definition 2: The range of shapeability is the closed set of values that the degree of shapeability may take, depending on the number of active inequality constraints.

B. Equality Constraints

The degree of shapeability of a system without inequality constraints can be determined based on the following.

Lemma 1: Degree of shapeability ($S$) of a system with only inequality constraints is equal to the minimum number of free parameters in any one row of the $\bar{\mathbf{A}}$ matrix.

A proof of this lemma is provided in the appendix. Clearly, the number of assignable eigenstates cannot exceed the total number of eigenstates for a system. Hence, the degree of shapeability $S$ cannot exceed the dimension $D$ of the $\bar{\mathbf{A}}$ matrix (i.e. $S \leq D$).

Practical application of Lemma 1 is straightforward if all equality constraints apply to individual parameters or to a linear combination of parameters in the same row. The number of free parameters in a row $i$ is equal to the dimension of the row $D_i$ reduced by the number of independent equality constraints $N_i$. Considering the most constraining row, the following is true if no equality constraints apply across rows of $\bar{\mathbf{A}}$. 
\[ S_\omega = D - \max_i (N_i) \]  

Here \( S_\omega \) is the degree of shapeability for a particular set of equality constraints. If all equality constraints apply either to individual elements of \( \tilde{A} \) or to multiple elements within a single row of \( \tilde{A} \) then no ambiguity exists in associating constraints with the rows of \( \tilde{A} \) and hence \( S = S_\omega \). However, when an equality constraint applies to a linear combination of elements across multiple rows of \( \tilde{A} \), then the constraint may be associated with any one of those rows. As such, there may be multiple ways of associating equality constraints with the rows of \( \tilde{A} \). The full set of allowable associations is defined to be \( \Omega \). Considering all possible associations \( (\omega \in \Omega) \), the degree of shapeability is based on the association that allows the largest number of eigenstates to be assigned.

\[ S = \max_{\omega \in \Omega} (S_\omega) \]  

The designer can choose to assign any number of eigenstates up to degree of shapeability \( S \).

A visual representation of the effect of shapeability on the solution of equation (7), for a system with no inequality constraints, can be seen in Figure 2. The figure illustrates the approximate Pareto optimal curves \( (J_1 \text{ as a function of } J_2) \) for a system with a variable number of actuators, much like the robotic system of Figure 1. As the number of actuators increases, the designer has more ability to specify the eigenstates of the dynamic system. Hence, the system is more shapeable with more actuators, and less shapeable with fewer actuators. When the number of actuators is not sufficient to achieve the desired eigenstates exactly, the value of \( J_1 \) cannot be set to zero. This limitation is represented in the figure when, as the number of available actuators decreases, the approximate Pareto optimal curve lifts away from the horizontal axis, indicating that \( J_1 \) is never zero and hence that an exact solution cannot be achieved. Even when an exact solution can be achieved (e.g. when the approximate Pareto optimal front intersects the horizontal axis with \( J_1 \) equal zero), additional actuators may still help in mitigating unspecified dynamics. In other words, additional actuators may aid in reducing the maximum unspecified eigenvalue \( J_2 \) for a given value of \( J_1 \). In the illustration, this effect can be identified by noting that the approximate Pareto fronts typically shift downward and to the left as the number of actuators is increased.

Figure 2: The effect of shapeability on the solution of the MEDFRD method; The zero location on the \( J_2 \) axis indicates the location of neutral stability.

When the degree of shapeability \( S \) becomes less than the number of specified eigenstates \( Q \), then an exact solution for the specified eigenstates is not possible. Any system in which \( Q > S \) will be referred as unshapeable since an exact solution to (7) is not feasible. Otherwise, the system will be referred to as shapeable.

C. Inequality Constraints

When the system has inequality constraints the degree of shapeability can reduce from that determined purely by equality constraints. A range of shapeability can be generated and is described by Corollary 1.

**Corollary 1:** The shapeability \( S \) of a system with inequality constraints is between \( \delta \) and 0, where the upper bound \( \delta \) is computed using equation (9).

This corollary is essentially a loose bound on the range of shapeability. The upper bound \( \delta \) is based on the notion that the number of free parameters in the \( \tilde{A} \) matrix can never be higher than in the case when no inequality constraint is active. In this limiting case, shapeability is set only by equality constraints, according to equation (9). As more inequality constraints become active, the number of free parameters in the system decreases, and shapeability can only decrease. In some cases, the number of inequality constraints may be sufficiently small that the lower end of the range of shapeability is a positive integer; however, it is likely that every parameter in the \( \tilde{A} \) matrix is subject to some practical engineering limitations (such that every element in the matrix is subject to at least one inequality constraint). In this limiting case, it is possible to specify certain eigenstates which will cause inequality constraints on all matrix parameters to become active, thus pushing shapeability to zero. For these reasons, it is justifiable to state that shapeability must lie between \( \delta \) and 0 for any system with inequality constraints.

Corollary 1 emphasizes that inequality constraints can have significant effects on the ability to specify eigenstates for a system. The degree of shapeability determined by equality constraints can reduce from that determined purely by equality constraints. If all equality constraints apply either to individual elements of \( \tilde{A} \) or to multiple elements within a single row of \( \tilde{A} \) then no ambiguity exists in associating constraints with the rows of \( \tilde{A} \) and hence \( S = S_\omega \). However, when an equality constraint applies to a linear combination of elements across multiple rows of \( \tilde{A} \), then the constraint may be associated with any one of those rows. As such, there may be multiple ways of associating equality constraints with the rows of \( \tilde{A} \). The full set of allowable associations is defined to be \( \Omega \). Considering all possible associations \( (\omega \in \Omega) \), the degree of shapeability is based on the association that allows the largest number of eigenstates to be assigned.

\[ S = \max_{\omega \in \Omega} (S_\omega) \]
constraints should not be over-interpreted. Because of inequality constraints, it may be possible to assign a particular eigenstate, but it still may not be possible to assign that state to a desired target value.

If inequality bounds on variables of $\tilde{A}$ are wide, then an exact solution to (7) may be possible for any shapeable system. However, as inequality constraints tighten, exact eigenstate specification can be made impossible, even for a fully shapeable system according to equality constraints, in which $\delta$ is equal to $D$. The addition of active inequality constraints effectively increases the number of equality constraints on the system, and can reduce the shapeability of a system. Typical Pareto optimal curves for tightened inequality constraints can be seen in Figure 3.

IV. APPLICATION TO THREE-LINK PENDULUM

A. System Model

In this section we investigate the utility of the degree of shapeability metric in characterizing a representative mechanical system: a feedback-actuated, three-link pendulum. The pendulum was selected as a simple mechanical analog to a robotic system (Figure 1), an analog featuring predominantly linear dynamics. In this section, the MEDFRD method is used to assign eigenstates to the three-link pendulum for several scenarios in which different numbers of joint-torque actuators are available and in which different sets of equality and inequality constraints are active. These results are compared to the degree of shapeability metric, to assess the utility of this metric in determining the number of actuators needed to realize a particular desired set of eigenstates. The three-link pendulum model can be seen in Figure 4.

![Three-link pendulum with variable passive dynamical parameters](image)

Figure 4: Three-link pendulum with variable passive dynamical parameters (length $L_i$, mass $M_i$, rotational spring constant $k_i$, and inertia $I_i$) and feedback-controlled joint torques ($T_i$).

For the three-link pendulum system the $\tilde{A}$ matrix is in $\mathbb{R}^{6 \times 6}$. Half of the rows of the $\tilde{A}$ matrix are kinematic equations; the other half of the rows are moment balances. The kinematic equations cannot be influenced through tuning of physical or feedback parameters; by contrast, tuning is possible using parameters in each of the moment balance equations. This distinction is made clear by writing out the general form of the linearized moment balance and kinematic equations for each link. For these equations $\theta_i$ represents the angle of $i^{th}$ link of the pendulum chain relative to vertical.

\[ C \frac{d}{dt} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = D \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + e \]  

(10)

\[ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \]  

(11)
The kinematic equations represented in (11) and do not include any physical or control parameters. The moment balance equations represented by (10) can be tuned with both physical and control parameters which are encompassed in the C and D matrices, and the e vector.

\[
C = \begin{bmatrix}
I_1 & I_1 M_2 \frac{I_2}{2} & I_1 M_3 \frac{I_3}{2} \\
0 & I_2 & I_2 M_3 \frac{I_3}{2} \\
0 & 0 & I_3
\end{bmatrix}
\] (12)

\[
D = \begin{bmatrix}
-M_1 g \frac{I_2}{2} - k_1 - k_2 & k_2 & 0 \\
k_2 & -M_2 g \frac{I_3}{2} - k_2 - k_3 & k_3 \\
0 & k_3 & -M_3 g \frac{I_3}{2} - k_3
\end{bmatrix}
\] (13)

\[
e = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} K z
\] (14)

The physical parameters of the links including mass, length, rotational spring constant, and inertia are represented respectively as \(M, L, k,\) and \(I.\) The matrix \(C\) is comprised of inertial, mass and length parameters. The same is true for matrix \(D,\) but with the addition of spring constant terms. The vector \(e\) represents the summation of joint torques acting on each link, and is comprised of a constant matrix, a matrix of control gains selected \(K,\) and the states of the system \(z\) (as can be seen in equation 15).

Using a state vector comprised of link angles and angular velocities, seen below, it is trivial to convert the dynamic equations (10) and (11) into a state-space representation of the system. The state vector is

\[
z = \begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 \\
\theta_1 \\
\theta_2 \\
\theta_3
\end{bmatrix}
\] (15)

The values of the first three rows of \(\bar{A}\) for the three-link pendulum with state vector (15) can be set to any arbitrary value, by varying design parameters, as is expressed in equation (10). There are no tunable parameters in the final three kinematic rows of the three-link pendulum’s \(\bar{A}\) matrix, from (11).

**B. Specified Eigenstate**

We wish to determine the effect of varying number of actuators and varying parameter constraints on attaining specified eigenstates. In order to do this we specified an oscillating mode shape for the three-link pendulum. In fact, to specify a single oscillating mode shape, we must specify two eigenstates which are complex conjugates. In our example, it was chosen arbitrarily that the links oscillate at a frequency of \(\pi/2\) rad/sec with a damping ratio of 0.063. The eigenvalue for this case is \(\lambda = 0.1 + \frac{2}{\sqrt{2}} j.\) Furthermore, it was chosen arbitrarily that the second link angle \(\theta_2\) should lag the first \(\theta_1\) by one-quarter of a cycle \((\pi/2 \text{ rad}),\) and that the third link angle \(\theta_3\) should lag the first by only 1/8 cycle \((\pi/4 \text{ rad}).\) The resulting eigenvector (accounting for coupling between angular velocity and angle) is

\[
x = \begin{bmatrix}
1 \\
(1 + j)/\sqrt{2} \\
\pi/2 \\
-\pi/2 j \\
\pi/2 (1 - j) /\sqrt{2}
\end{bmatrix}
\] (16)

It must be noted that in order to obtain a periodic robot gait, it is necessary to perturb the system by inputting a periodic forcing function (for stable of marginally stable systems) or by entraining a limit cycle (for unstable systems) [13].

The complementary eigenvalue and eigenvector (of the oscillatory pair) were specified as the complex conjugates of the primary eigenvalue \(\lambda\) and the primary eigenvector \(x\) described by (16). Hence, for this system, the total number of specified eigenstates \(Q\) is two. According to shapeability analysis, we expect that system shapeability \(S\) must be at least two in order to specify these eigenstates exactly.

**C. Variable Degree of Shapeability and Constraints**

We first consider the eigenstate specification problem for a series of progressively less actuated scenarios, in which three, two, one, or zero joint torques can be commanded via feedback control. More specifically, removal of joint torques progresses downward from the constrained end of the three-link chain. As joint torques are removed, the associated rows of \(\bar{A}\) have fewer tunable parameters as more equality constraints become active. Equality constraints are introduced when an element of the \(\bar{A}\) matrix equals zero. As actuation is removed, some elements of the \(\bar{A}\) matrix can no longer be shifted by feedback and hence become constrained to equal zero. This removal of torques causes the system’s degree of shapeability to change. For designs with a smaller number of actuators, the number of free variables in \(\bar{A}\) decreases. When the shapeability \(S\) is smaller than the specified number of eigenstates \(Q,\) it is anticipated that the system is unshapeable, and hence the desired eigenstates cannot be obtained exactly.

This test will be repeated twice, each time considering a different set of inequality constraints on the system’s design parameters. The inequality constraints were arbitrarily selected to demonstrate contrasting extremes of constraint scenarios and demonstrate the range of shapeability concept. In the first case, the system parameters are only very loosely bounded, such that inequality constraints play only a very small role in eigenstate assignment. In this case, we expect the
shapeability to approach its upper limit ($S \rightarrow \delta$). In the second case, the system parameters are much more tightly bounded, such that inequality constraints significantly restrict eigenstate assignment, effectively reducing the degree of shapeability and causing eigenstate specification to have greater errors. In other words, the system’s shapeability approaches its lower limit ($S \rightarrow 0$) as constraints tighten. A comparison of the two cases emphasizes the role that inequality constraints play in assigning a system’s eigenstates. The constraints used in each of the two cases are summarized in Table I.

**TABLE I: Inequality constraints on the $\tilde{A}$ matrix.**

<table>
<thead>
<tr>
<th>Tight Bounds</th>
<th>Loose Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>Length</td>
</tr>
<tr>
<td>[0.01-0.05]</td>
<td>[0.01-0.05]</td>
</tr>
<tr>
<td>Mass</td>
<td>Length</td>
</tr>
<tr>
<td>[0-inf]</td>
<td>[0-inf]</td>
</tr>
</tbody>
</table>

V. RESULTS

To characterize the three-link pendulum system described in the previous section, the MEDFRD method was applied to each of eight instances: two cases of different inequality constraints on design parameters (Table I), each featuring four scenarios with different numbers of joint-torque actuators available (from three down to zero). Each of these eight instances is best characterized by a set of results (an approximate Pareto optimal front) and not by a single result. To provide a simpler basis for comparison, however, we considered only a single point for each approximate Pareto optimal curve, the point where the value of $J_2$ was set to zero. By holding $J_2$ constant the degree of shapeability and inequality constraints could be analyzed independent of the unspecified dynamics.

In solving equation (7) with MEDFRD, a genetic algorithm was used rather than a gradient descent method because the cost function $J_2$ is not convex. The results of these computations are summarized in Table II. Values of $J_1$ for tests in which the loose bounds were applied are in the column labeled “Loose.” Values of $J_1$ when tight bounds were applied are in the column labeled “Tight.”

**TABLE II: Varying Constraints and Shapeability.**

<table>
<thead>
<tr>
<th>Actuators</th>
<th>$\delta$</th>
<th>$J_1$ (Loose)</th>
<th>$J_1$ (Tight)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.032</td>
<td>0.84</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0.036</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The first and last lines of Table II represent two extremes of eigenstate assignment; these extremes are covered by existing methods, which predated MEDFRD. The first line of Table II corresponds to the scenario in which no actuators are present.

In this case, dynamics are purely passive, and so eigenstate placement would be possible using passive dynamic methods [3]. The last line of Table II corresponds to the scenario in which all joints are actuated. In this scenario, it is always possible to satisfy equation (3) exactly using modal control [8-11], so long as the pendulum angular velocities are kinematically consistent with the pendulum angles. The two middle lines of Table II are particularly interesting in that they exercise the capabilities of MEDFRD, allowing for exact or approximate eigenstate assignment, even for underactuated systems. In this sense, MEDFRD bridges the gap between passive parameter tuning and conventional modal control, covering both extremes as well as the range in between.

As actuators are removed from the three-link pendulum system, the degree of shapeability of the system changes. The upper bound on the degree of shapeability $\delta$ for each scenario is listed in the second column of Table II. In the Loose Constraint case, it is immediately evident that the shapeability limit indicates whether or not an exact solution is possible. When $\delta$ is larger than the number of eigenstates specified ($Q = 2$), in the second and subsequent rows of the table, then the value of the cost function $J_1$ can be pushed to its minimum value of zero, which indicates the desired eigenvalues are obtained exactly. When $\delta$ is less than $Q$, as occurs in the first line of the table, then an exact solution is not feasible. Thus, when inequality constraints are loose, degree of shapeability provides a clear indication of whether or not an exact solution can be computed.

The role of $\delta$ is less crisp when inequality constraints are tight. In effect, the tight inequality constraints make the feasible space of eigenstates smaller and smaller, until their distinction from equality constraints blurs. For the case of tight inequality constraints, the only exact solution was achieved by the fully actuated system (last line of Table II). The fully actuated system has a shapeability equal to the dimension of the $\tilde{A}$ matrix ($\delta = D$), and since there are a large number of control parameters available for tuning, the inequality constraints have relatively little impact on system design. For all scenarios with two or fewer actuators, an exact solution was not possible with the tight inequality constraints. This limitation was true even for cases in which $\delta$ was greater than $Q$. Thus, in the case of tight inequality constraints, the $\delta$ metric generated by equality constraints provides only partial information about whether or not a particular eigenstate can be assigned to a dynamic system. Effectively, the degree of shapeability was reduced toward zero, as expected in the case of tight inequality constraints. Nonetheless, the upper bound $\delta$, nominally associated with equality constraints, was still indicative of the quality of the approximate solution, even in the case of tight inequality constraints. Specifically, a better approximation was achieved for partially actuated systems when $\delta$ was larger.

VI. CONCLUSION

This paper introduced a new metric for determining the number of eigenstates which can be specified for a linear
dynamic system. This metric, called degree of shapeability, provides insight into eigenstate specification by considering both the actuation, morphology, and design requirements for the system. Degree of shapeability is particularly important in analyzing underactuated cases (such as the three-link pendulum example with only one or two active joint-torque actuators). In concept, a designer could use the degree of shapeability metric, rather than running a full design method (such as MEDFORD), to provide a preliminary indication of the number of actuators needed to achieve a desired set of eigenstates. Degree of shapeability is defined crisply (at the upper limit 6) when only equality constraints are present. When inequality constraints are present in a systems \( A \) matrix, the degree of shapeability is confined to a closed set of values called the range of shapeability.

REFERENCES


APPENDIX

Lemma 1: Degree of shapeability is equal to the minimum number of free parameters in any one row of the \( A \) matrix.

Proof: First consider the case in which only one eigenstate is assigned. If the eigenstate consists of the eigenvalue \( \lambda_0 \) and the eigenvector \( x_a \), then the following equation must be true.

\[
\tilde{A} x_a = \lambda_a x_a
\] (17)

Each row of this vector equation may be written as follows, where \( \tilde{A}_i \) is the \( i^{th} \) row of the \( A \) matrix and where \( x_{a_i} \) is the \( i^{th} \) element of the \( x_a \) vector.

\[
\tilde{A}_i x_a = \lambda_a x_{a_i}
\] (18)

In order to satisfy equation (18) for each \( i \), given that the eigenvector and eigenvalue are fixed, at least one element of each row vector \( \tilde{A}_i \) must be a free parameter. Hence, specifying one arbitrary eigenstate for a system requires that at least one element in each row of \( A \) is a free parameter.

Next consider the case when a second arbitrary eigenstate is specified (consisting of eigenvalue \( \lambda_b \) and eigenvector \( x_b \)). Applying (3) for each eigenstate, the following must be true.

\[
\tilde{A}[x_a \ x_b] = [x_a \ x_b] \left[ \begin{array}{c} \lambda_a \\ 0 \\ \lambda_b \end{array} \right]
\] (19)

This equation puts two constraints on each row of the \( A \) matrix.

\[
\tilde{A}_i x_a = \lambda_a x_{a_i} \\
\tilde{A}_i x_b = \lambda_b x_{b_i}
\] (20)

Equation (20) represents a set of two linearly independent constraints on the elements of \( \tilde{A}_i \). This is true because the set of eigenvectors must be linearly independent for an eigenvalue transformation to exist. Hence, the two eigenvectors are linearly independent and so are the two constraints of (20). To satisfy the two constraints of (20), a solution is only possible if \( \tilde{A}_i \) contains two free parameters. Hence, specifying two eigenstates for a system requires that at least two elements in each row of the transition matrix \( A \) are free parameters.

An extension of this logic makes possible a proof by induction. Consider the case in which \( S \) eigenstates are specified. The generalization of (20) places \( S \) linearly independent constraints on each row vector \( \tilde{A}_i \). Hence, specifying \( S \) eigenstates for a system requires at least \( S \) free parameters in each row of \( A \).