Development of Robust Crawling in Soft-Bodied Robots

A thesis submitted by

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Abstract

Soft-bodied robotics is a new class of robotic technology defined by high deformability; an attribute which promises to extend the reach of current robotics. Navigation through constricted passages is a particularly challenging task for modern robots because their geometry dictates a specific envelope through which they can move. Soft-bodied robots, not possessing specific geometry, are able to “spread out” in one dimension in order to make another dimension smaller. To further the development of soft robotics this thesis presents a method for disturbance rejection for the establishment of robust gaits and a template for soft-bodied locomotion which is useable in the creation of future soft bodied systems by acting as a prediction model of system energy transfer. These two contributions are a start towards efficient, stable, predictable locomotion in mobile soft robotic systems.
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# Table of Contents

1 Introduction 
   1.1 Motivation 1 
   1.2 Background 3 
   1.3 Contributions 8 
   1.4 Guide 10 

2 Generation of Limit Cycles in Hybrid Systems Consisting of Two Linear Subsystems 12 
   2.1 Nomenclature 12 
   2.2 Introduction 12 
   2.3 Definitions 15 
   2.4 System Description 15 
   2.5 Method Development 17 
      2.5.1 General Limit Cycle Properties 18 
      2.5.2 Specification of Bulk Properties 27 
   2.6 Design Procedure 28 
   2.7 Example Case 29 
   2.8 Conclusion 30 

3 Template for Robust Soft-Body Crawling with Reflex-Triggered Gripping 31 
   3.1 Introduction 31 
   3.2 Physics-Based Template 35 
      3.2.1 Fundamental Concepts 36 
      3.2.2 Coupled Differential Equations 39 
   3.3 Control Method 43 
   3.4 Methods 46 
      3.4.1 Simulation Success and Failure 47 
      3.4.2 Distance Traveled 49 
   3.5 Results 49 
      3.5.1 Qualitative Gait Comparison 49 
      3.5.2 Sensitivity Results 52 
   3.6 Discussion 54
3.6.1 Mathematical Robustness 55  
3.6.2 Physical Robustness 56  
3.6.3 Biological Robustness 58  
3.7 Conclusion 58  
4 Thesis Conclusion 60  
 4.1 Contributions 60  
 4.2 Future Work 63  
 4.3 Technical Impact 64  
A Appendix – Background Mathematics 68  
  A.1.1 Control of Linear Systems 68  
  A.1.2 Hybrid Systems 75  
B Bibliography 81
Figure 1-1 - Educational toy or game apparatus. US 741,903 A .................................................. 3
Figure 1-2 – Side view of such a motor setup with (top) a rigid rod and (bottom) soft noodle. .... 4
Figure 1-3 – Top view of a hard (H) and soft (S) rod being rotated by a motor. ......................... 5
Figure 1-4 - A passive dynamic walking robot (Energy Efficient Robotic System US
20100170729 A1) ......................................................................................................................... 7
Figure 2-1 - Schematic of proposed hybrid system ................................................................. 17
Figure 2-2 - Phase portrait of nominal trajectory............................................................ 19
Figure 2-3 - Representative phase portrait of trajectory defining $f(m)$ ............................ 20
Figure 2-4 - Representative phase portrait of trajectory defining $g(m)$ ............................ 21
Figure 2-5 - Plot of functions and their composition which describe a representative iterated
functional system .......................................................................................................................... 22
Figure 3-1. Overview of the model showing the lumped mass body segments connected to each
other. ........................................................................................................................................ 37
Figure 3-2. A single body segment schematic showing the spring, damper, and muscle
attachments .............................................................................................................................. 37
Figure 3-3. Diagram of gravity vector orientation ($\theta = 0$ is horizontal crawl) .................. 38
Figure 3-4. Representation of rigid/compressible bodies in relation to triangle inequality....... 45
Figure 3-5. Hybrid system description for the crochet (left) and muscle (right) activation. .... 46
Figure 3-6. Comparison of Crawl Timing for $M. sexta$ and Model......................................... 50
Figure 3-7. (left) Vertical displacement of prolegs during a crawl (adapted from (Trimmer and
Issberner 2007)). (right) Vertical displacement of body segments interpolated from segment
length and crochet attachment positions with applied loess smoothing .................................. 52
Figure 3-8. Representative Dual-Parameter Sensitivity Result ............................................. 53
Figure A-1- Phase portraits of systems exhibiting various categories of eigenvalues ............ 73
Figure A-2- Hybrid system of a ball in flight ........................................................................ 78

Table 2-1 - Design values and Bulk Properties for Example Configuration 30
Table 3-1 Nominal parameter values for simulation 47
Table 3-2 Single Parameter Sensitivity 52
1 Introduction

1.1 Motivation

Since the industrial revolution the face of labor has been changing. Craftsmen were replaced by the factory, drastically improving productivity (Deane 1965). A second revolution is now upon us: robotics. The employment of industrial robots has expanded the production capacity drastically due to the ability of robots to work tirelessly and produce things faster and more accurately than the humans that they replaced (Rifkin 1995). These industrial robots are designed for the environment they work in which means that they are able to be firmly rooted to the factory floor and use large actuators to move the heavy rigid beams comprising their “body”. Outside of the factory this design style finds far less utility. Robots which operate in the world beyond the factory walls must be able to locomote through the wide variety of terrains the world has to offer (Nehmzow 2003). Not being tied to a specific spot introduces the additional complication of not being able to receive a constant source of energy. In order to truly take advantage of free-range robots they will have to overcome these obstacles and go beyond to do things that humans are not able to do. The next generation of robots will succeed because of the development of new capabilities, novel structures, and unique controls.

Industrial robots have found homes in a wide variety of applications: welding, painting, assembly, pick-and-place, and numerous others. Each of these applications is benefitted by the advantageous aspects of rigid robots. Because they consist of a series of links of known length it is trivial to perform the calculations required to determine the position of the end effector (Paul
However, this knowledge comes at a price. To get to all of the weld locations in a car, for example, the rigid links need to be accommodated by the car. This means openings in the frame need to be wide and in consistent locations. In the environment of the factory these factors can be controlled.

Outside of the factory methods of locomotion have to be considered. Mobile robots have already been utilized in bomb disposal, driverless cars, and surveillance. Current mobile robots use wheels or legs to facilitate their travels (Sugiyama and Hirai 2006), but these rigid constructs mean that they are limited to relatively open areas. Traveling through dense brush and fallen logs without leaving a swath of destruction is beyond their capacity.

Being able to construct robots that can travel through terrain defined by long, tight passages of uncertain geometry would prove useful in such circumstances as those that present themselves in the aftermath of disasters where collapse has rendered once open pathways into cramped channels constricted by rubble. Consider the aftermath of the September 11th terrorist attacks. Numerous people were trapped in voids inaccessible by humans or the robots of the time. Communication and provisions are important to keeping trapped people alive and eventually rescuing them. Even in the wreckage there were passages, but the shape would not be consistent across passages or throughout the entirety of a single one. Rigid links have a specific shape meaning that they could only fit into holes which accommodate their unyielding geometry.
Soft robots, on the other hand, are able to deform in all sorts of ways (Umedachi 2013) meaning they are unencumbered by changing pathway size and shape. Soft robots are able to stretch in one dimension to reduce their size in another (Keller 1983). They don’t have a defined cross-section which allows them to maneuver through holes of a wide range of aspect ratios. Such a robot would be able to wriggle its way through a series of contortions into otherwise inaccessible areas. However, such robots are not as simply controlled as those constructed of rigid links. Without a good understanding of how the robot will react to stimuli it is impossible to come up with a method for providing the right stimulus to elicit the desired response (Walker and al. 2005). The motion of rigid robots is understood through classical geometry, whereas continuously deformable robots will require more modern methods.

1.2 Background

Soft robotics is still a topic of research very much in its infancy (Kim, Laschi and and Trimmer 2014). There is currently boundless promise, but the actual implementation of these ideas is one
that has been slow in coming to fruition. The progression towards soft robots has followed a path of increasing degrees of freedom in rigid robots in an attempt to reach some breaking point where the benefits of soft robots will present themselves (Chen, Ma and et. al. 2007). Such highly articulated robots have been studied in-depth under the banner of “snake-like” robots (Chen, Ma and et. al. 2007) (Date and Takita 2007). As the number of segments increases, however, new challenges are presented (Walker and al. 2005). No matter what direction is explored there are obvious gaps which need to be overcome before soft robots are a reality. It will only be through a concerted effort of a variety of disciplines that all of the difficulties will find solutions.

Even the simplest tasks that are expected of a robot are difficult to implement given the current state of soft robotics. To gain an appreciation for these problems consider a motor with two potential attachments: a steel rod and a limp spaghetti noodle. How these two end effectors interact with their environment is totally different. The steel rod will stay reasonably planar with gravity as long as it is not too long. The noodle, on the other hand, may droop to the ground and experience a whole new environment.

![Figure 1-2](image)

Figure 1-2 – Side view of such a motor setup with (top) a rigid rod and (bottom) soft noodle. Rigid parts are able to maintain their geometry under their own weight whereas soft objects collapse.
It is easy to see how rotating the motor by ninety degrees results in a ninety degree deflection of the end of the steel rod. A similar actuation with the noodle attachment has wildly different results. After the rotation the noodle will take up some curve which approximates the ninety degree rotation locally to the actuator but which may leave the end farthest away unmoved due to frictional interactions with the ground.

![Figure 1-3 – Top view of a hard (H) and soft (S) rod being rotated by a motor. Since the soft rod is in contact with the ground and able is effected by friction it will obtain a bent shape whereas the rigid rod attains a very predictable position](image)

This curve may also be a function of the velocity at which the motor rotated: another variable that now needs to be considered. How the noodle finally comes to rest is also dependent on the initial configuration. Bit by bit the complications of moving a soft object become apparent. For soft robots to locomote there is going to have to be a way to account for or overcome all of these effects.

Part of the process towards achieving soft robotic locomotion is designing control systems which reduce the nearly countless variables to something more manageable. One potential direction is to build systems which are able to reject many of the smaller effects.
allowing the designer to focus on a more limited set of variables. This can take the form of reducing the continuous system to a set of discrete lumped masses (Chen, Ma and et. al. 2007) (Rincon and Sotelo 2003) or creating a control system which naturally accounts for some variability. An example of the second type of control scheme is known as a limit cycle (Flieller, Riedinger and Louis 2006). In a limit cycle there is some nominal path of position and velocity that the robot will cycle through forever, and if something causes the robot to deviate from that path the system will track back towards it (Chemori and Alamir 2005). In a world of unexpected disruptions having a system that operates on a limit cycle is very useful (Goswami, Espiau and Keramane 1997).

Another potential direction is to create a system with a built-in way of reducing its degrees of freedom (Yekutieli, et al. 2005). As with the noodle, effects from actuators are typically rather tractable near the location on which they are acting. A soft robot with the ability to tie itself to the ground at various points throughout its length would have less variability in its position and begin to allow for more informed control of the free segments. All that is left then is to coordinate which parts of the robot are free and which are strapped to the ground.

This second approach is exactly what is seen in the tobacco horned-worm (Manduca sexta) as it travels through the vines and branches that make up its home. A system designed with inspiration from them could be able to duplicate such intricate motions through the equally complex terrain present in a collapsed building (Kate and al. 2008). Segmenting a soft, continuous system into discrete blocks is exactly the approach that is investigated in this thesis. M. sexta is not only able to crawl about in a convoluted environment (Trimmer. and al. 2006) but it does so with a very limited range of sensors and an easily traceable system of actuation.
This makes it an ideal candidate for a model of a soft, low-cost robot able to navigate a tangle of twisted metal and concrete.

Using nature for inspiration is nothing new (Menon and Setti 2006), and has in fact shown to be a very effective method of producing apparatuses exhibiting desired behaviors. Animals can be viewed as working prototypes, the properties of which can be coopted into future designs (Pfeifer 2007). For example the quest for efficient locomotion has given rise to the passive dynamic walker, a human inspired robot that requires nothing more than a slight inclined to produce motion. An early developer in this area, Dr. Tad McGeer, noticed that humans swing their arms to store energy during one part of each step and release it during another and was able to abstract this concept and then apply it to the robot (McGeer 1990). Abstraction from a human walking to an understanding of the energy transfers involved can be understood through the development of a model called a *template* (Full and Koditschek, Templates and Anchors: Neuromechanical Hypotheses of Legged Locomotion on Land 1999). These are reduced-order dynamic models which account for the fundamental aspects of the locomotive process but are general enough to be used to describe locomotion in a wide range of organisms.

*Figure 1-4 - A passive dynamic walking robot (Energy Efficient Robotic System US 20100170729 A1)*
Not only are walkers efficient but they are able to reject small disturbances through their design. While these devices have many nice characteristics they are still little more than academic curiosities because of the limitations to the environments in which they can operate. Hopefully by investigating other biological systems a future robot will be able to incorporate some of the benefits of the passive dynamic walkers while having the ability to work in a wide range of situations.

1.3 Contributions

This thesis has two main sections aimed at furthering different aspect of mobile robotics. The first investigates a novel way of controlling a system such that it follows a limit cycle. What is more, the proposed method not only produces a limit cycle but allows the designer to specify (to a degree) the particular path that makes up this limit cycle. The second part aims to start to solve the control problems associated with soft systems by looking at natural systems that display the desired properties. Through the use of a paired down model of *M. sexta* it demonstrates a possible control strategy that produces forward motion.

More specifically the contributions are:

(Chapter 2 – Development of Limit Cycles in a Class of Hybrid Systems Consisting of Two Linear Subsystems)

- Development of a procedure to generate a control system which produces a stable output of a desired amplitude and frequency. A designer with a linear system model and the desire for this system to be entrained into a limit cycle with a specific period and amplitude will be able to utilize the design method proposed here to achieve their goal.
Proving the stability of this system mathematically. Starting from basic principles of linear systems this thesis proves that a limit cycle with the desired properties exists. Since it works with a very general system this approach will be able to be expanded to work with specific systems.

Verifying, through simulation, that the design procedure works. A numerical example is run in order to show how period and amplitude can be specified. Even designers without an understanding of the details of the stability proof will be able to follow the procedure carried out in the example.

(Chapter 3 - Template for Robust Soft-Body Crawling with Reflex-Triggered Gripping)

- Generation of a control system for a class of soft robots, specifically those employing grippers to attach to a substrate. This thesis develops a soft model of a biological specimen using lumped masses connected by springs, dampers, and actuators then uses this model to show the efficacy of a proposed gripper control method. Importantly, the control method utilizes only sensor data representative of that available in biological systems. This control system is useful to the study of soft robotics because it is distributed and free of hierarchy.

- Characterization of the robustness of the template. The model proposed is shown to be insensitive to a wide variety of parameters. This result provides a theory on how *M. sexta* is able to maintain a similar gait throughout its growth and development.

These contributions are a modest first step towards the development of soft robotic systems able to perform all of the actions that nature has shown are possible. Future roboticists can build on these developments to create mechanisms that explore places previously inaccessible.
1.4 Guide

The overarching theme of this thesis is that soft robots (if properly controlled) will have access to previously impervious environments. To this end research is presented which explores fundamental concepts regarding how this control can be implemented.

Following this introduction, the main body of the thesis consists of two papers offered up to peer review. The first paper (Chapter 2 in this thesis) was accepted to the 2014 American Controls Conference. Entitled “Generation of Limit Cycles in Hybrid Systems Consisting of Two Linear Subsystems,” this paper was written to explain how hybrid systems break linearity in a way that can be exploited to produce limit cycles. As stated previously limit cycles are advantageous because they seek to maintain their nominal trajectory, effectively rejecting disturbances. Being able to traverse complex environments speaks directly to the theme. The second paper (Chapter 3 in this thesis) has been submitted at the journal Bioinspiration and Biomimetics. The submission was titled “Template for Robust Soft-Body Crawling with Reflex-Triggered Gripping.” Here a generalization of soft-body crawling is explored in the context of how we can adopt methods employed by nature to engineer robotic systems with similar capabilities.

These two chapters come together to provide some insight into how to control systems in a robust way. While a real caterpillar is far more complex than the template presented in Chapter 3 this model captures very important features of caterpillar locomotion. Notably, this model has continuous dynamics in the form of a second order system. The limit cycle investigation in Chapter 2 is also a second order system which may provide the compatibility necessary for them to be combined. A fusion of the two ideas into a comprehensive control scheme would have the ability to extend the region of stability of a soft system employing the template presented. Even
if a complete synthesis is not possible the fact that they both operate on a similar class of systems means that they could be used independently on the same platform to grant their particular stability in areas which need one in particular.

This thesis is rounded out with a conclusion and a brief treatment of the mathematical topics relevant to linear hybrid systems. The conclusion that details the important aspects that have been presented and provides a look at some potential directions that the author sees for future research.

An appendix provides a broad introduction to the mathematical theory used in the thesis, with an emphasis on linear systems (especially the state-space formulation) and hybrid systems.
2 Generation of Limit Cycles in Hybrid Systems Consisting of Two Linear Subsystems

2.1 Nomenclature

\( A \in \mathbb{R}^{2 \times 2} \)  The state matrix for the system being controlled
\( B \in \mathbb{R}^{2 \times 1} \)  The input matrix for the system being controlled
\( K_s \in \mathbb{R}^{1 \times 2} \)  A vector of control gains which makes the system stable
\( K_u \in \mathbb{R}^{1 \times 2} \)  A vector of control gains which makes the system unstable
\( x \in \mathbb{R}^{2 \times 1} \)  Vector of states \([x_1; x_2]\) where \( x_2 \) is the derivative of \( x_1 \).

\( s \)  The positive \( x_1 \) value which defines the positive switching surface.

\( t \)  Continuous time variable

\( Q=\{q_1, \ldots, q_n\} \)  Collection of discrete state variables

\( E=\{e_1, \ldots, e_i\} \)  Set of transition edges between discrete states

\( T=\{t_1, \ldots, t_i\} \)  Set of transition times between discrete states

\( \lambda = \alpha + i\beta \)  Generic eigenvalue

\( \lambda_s = \alpha_s \pm i\beta_s \)  Eigenvalues of the matrix \( A - BK_s \)

\( \lambda_u = \alpha_u \pm i\beta_u \)  Eigenvalues of the matrix \( A - BK_u \)

2.2 Introduction

Oscillating trajectories that are also limit cycles are of particular importance in the control of systems having a desired amplitude and frequency, such as human walking models (Goswami, Espiau and Keramane 1997). Limit cycles are able to reject disturbances without explicit trajectory tracking. In marginally stable linear systems, by contrast, oscillation amplitude is
defined by the initial conditions. If energy is added or removed from the marginally stable system, the steady-state sustained oscillation amplitude will change. Previous work has been done to show that periodic injection of energy can sustain oscillation in linear systems (Chemori and Alamir 2005). Being able to control general systems (even those with smooth vector fields) such that they have trajectories which are stable limit cycles is still an open (Pérez and Benítez 2011).

This thesis section describes a way of creating limit cycles from linear systems using a hybrid controller that forces the system to operate with alternating stable and unstable dynamics. Our particular design approach is to apply the method of pole placement to design full-state feedback controllers for distinct regions of a state space divided by switching surfaces. Within each region full-state feedback can be used to produce systems with stable, marginally stable, or unstable dynamics. By stitching together stabilizing and destabilizing control laws we arrive at a stable trajectory with all of the properties of a limit cycle.

Our work builds on prior research on stitching together linear systems to produce limit cycles. For example, Goncalves (Goncalves 2005) provides the conditions necessary for a piecewise linear system (PLS) to exhibit the properties of a limit cycle. Others have looked at how to detect the presence and stability of limit cycles in hybrid systems (Flieller, Riedinger and Louis 2006). These concepts have been leveraged to design hybrid controllers that achieve desirable trajectories. These desirable trajectories may be stabilizing, as in the case of Xu and Antsklis (Xu and Antsaklis 2000), who found systems governed by one of two unstable modes at any given time can be stabilized by selectively switching between the two modes according to a switching rule constructed to leverage the topological properties of the modes within each region. These desirable trajectories may also be limit cycles, as in the case of Perez and Benitez
(Pérez and Benítez 2011), who applied switching laws to non-linear systems and found some cases in which the stable solution was a limit cycle. Although these methods provide powerful descriptive tools for analyzing and synthesizing limit cycles, they do not provide tools for tuning the properties of those limit cycles.

Relatively little prior work has looked specifically at designing limit cycles with desirable properties. One specific example is the work of Kai and Masuda (Kai and Masuda 2012), who created switching surfaces and associated regional dynamics to construct very complex stable limit cycles working with affine systems. Another example is the work of Chemori and Alamir (Chemori and Alamir 2005), who described model-based approaches that leverage a large amount of computation to calculate the needed feedback to achieve very exact behaviors. Additional research has been done in constructing Lyapunov surfaces that will generate desired limit cycle trajectories (Adachi, Ushio and Yamamoto 2004). These methods provide powerful tools for developing precise limit-cycle trajectories; however, these approaches also have limitations. For instance, (Kai and Masuda 2012) does not obey the kinematic constraints of realizable, physical systems (e.g., the two states of a simple harmonic oscillator are decoupled such that $x_2$ is no longer the derivative of $x_1$). Other methods such as (Chemori and Alamir 2005) are very computationally expensive, using iterative numerics to simulate the system time response and determine what gains will produce the desired trajectory.

Our proposed alternative method seeks to simplify the design problem by specifying only bulk properties, such as amplitude and period, rather than detailed trajectories. Accordingly, we apply a hybrid controller with a very simple switching surface to a realizable linear system. The simplicity of the approach reduces the calculations needed to determine where switching should occur. Thus, our method has potential to aid designers to create limit-cycle based motion
patterns, for example to enable locomotion of robotic devices where energy consumption is a foremost concern.

The simplicity of the switching surface specification and its direct connection to these bulk properties means that it is easy and intuitive for an operator to change these properties during operation. Changing the location of the switching surface can be performed by altering a single setpoint and allows the operator to vary amplitude of the limit cycle smoothly. Other methods for limit cycle specification rely on the specification of abstract quantities which do not have such immediate connections to plant response.

2.3 Definitions

*Limit cycle* - a closed trajectory \( C \) on the phase plane such that there exists a finite time \( T \) such that \( C(t) = C(T + t) \) for all times \( t \) which is the limit set of trajectories starting in some neighborhood of \( C \) as time goes to positive or negative infinity. If all points in the neighborhood of \( C \) for which \( C \) is a limit set converge to that limit as time approaches positive infinity then the limit cycle is stable.

*Amplitude* – in steady-state the amplitude is the largest magnitude displacement along the \( x_1 \) axis that the system obtains.

*Frequency* – the number of closed loop circuits completed by the system per unit time.

2.4 System Description

Consider the second order linear system

\[
\ddot{x} + a\dot{x} + bx = u(t).
\]  

(2-1)

where \( a, b \in \mathbb{R} \). This can be represented as a set of first order linear equations by
\[
\dot{x} = Ax + Bu
\]  \hspace{1cm} (2-2)

In the case of kinematically constrained systems, such as the second-order system (2-1), it is not possible to control position directly as \( B = [0 \hspace{0.5cm} 1]^T \). Assuming a linear controller \( u = -Kx \), the state-space form of (2-2) is thus constrained to be the following.

\[
\dot{x} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ a_{cl} & b_{cl} \end{bmatrix} x
\]  \hspace{1cm} (2-3)

Here the parameters \( a_{cl}, b_{cl} \in \mathbb{R} \) are closed-loop analogues to the scalar parameters \( a \) and \( b \) from (2-1).

For a \( \lambda \in \{ \lambda_1, \lambda_2 | \lambda_1 = \bar{\lambda}_2 \} \) it is known that careful selection of the gain matrix \( K \) permits the identity:

\[
\det((A - BK) - \lambda I) = 0
\]  \hspace{1cm} (2-4)

If the real part of \( \lambda \) is positive then the system is unstable, and if it’s negative then the system is stable.

For this method we will consider a hybrid controller with two different gain vectors \( K \) defined over different regions of the state space. The two gain vectors are labeled \( K_u \) and \( K_s \), where the subscript indicates an oscillatory response that is unstable in one region (\( u \) subscript) and stable in the other (\( s \) subscript). This means that for \( \lambda \) of the form \( \alpha + \beta i \) we have \( \alpha_u > 0 \), \( \alpha_s < 0 \), and \( \beta_u, \beta_s > 0 \). This will be assumed throughout the rest of this thesis section.

The hybrid control law combines the stable and unstable gain vectors to produce the following piecewise definition of the system of linear ODEs describing the system dynamics.

\[
\dot{x} = \begin{cases} 
(A - BK_s)x, & q = 'STABLE' \\
(A - BK_u)x, & q = 'UNSTABLE'
\end{cases}
\]  \hspace{1cm} (2-5)
These dynamics depend on a discrete state $q \in Q$. The set of possible discrete state variables is binary: $Q = \{\text{STABLE, UNSTABLE}\}$. The set of transitions consist of $E = \{(\text{STABLE, UNSTABLE}), (\text{UNSTABLE, STABLE})\}$ with the switching rule set $S = \{|x_1| < s, |x_1| > s\}$.

The complete specification of the hybrid system is then given by the collection of discrete state variables ($Q$), the collection of continuous state variables ($x$), the equations of continuous dynamics in each discrete state (2-5), the transition ($E$), and the switching rule ($S$). The hybrid-control design problem seeks to obtain a limit cycle with desired properties through careful selection of the gains vectors ($K_u, K_s$) and of the switching surface location $s$.

![Schematic of proposed hybrid system](image)

The idea of being unstable in a region surrounding the origin and stable far away is also seen in the van der Pol oscillator (Khalil 2001) and other systems. The approach presented here borrows from this concept but gives a designer additional tuning parameters which can be used to control the output waveform.

### 2.5 Method Development

The proofs in this section are set up to provide an idea of how systems with these particular dynamics behave. Theorem 2-1 establishes that a nominal trajectory exists. Theorem
2-2 concerns the presence of a limit cycle.

### 2.5.1 General Limit Cycle Properties

*Theorem 2-1* – Given a switching surface $s$ a closed nominal trajectory can be constructed through the arbitrary selection of either $\lambda_u$ or $\lambda_s$.

**Proof:** Consider a switching surface defined by $|x_1| = s$. Choose initial condition $P_1 = (-s, m)$ for arbitrary $m > 0$. Assume the rotational direction of the system is such that the system is about to undergo transition to $q = UNSTABLE$. Choose $\lambda_u$ arbitrary. The solution to the ODE of this unstable system is:

$$
x(t) = 
\begin{bmatrix}
 e^{\alpha_ut} \left( \frac{m}{\beta} \sin \beta t - s \cos \beta t \right)

 - e^{\alpha_ut} \left[ (m \cos \beta t - s \beta \sin \beta t) + \right]

 \frac{m}{\alpha_u(\frac{m}{\beta} \sin \beta t + s \cos \beta t)}
\end{bmatrix}

(2-6)

As a limiting case, first consider a system which also starts at $P_1$ but for which $\lambda = (0 \pm i\beta)$. The solution to the ODE for this limiting case is:

$$
\bar{x}(t) = 
\begin{bmatrix}
 -\frac{m}{\beta} \sin \beta t - s \cos \beta t

 -m \cos \beta t + s \beta \sin \beta t
\end{bmatrix}

(2-7)

For the limiting case, the next transition across a switching surface will occur when $\bar{x}(t) = (s, m)$. For the more general case, $\lambda$ is unstable and $x(t)$ will undergo transition at $P_2 = (s, n)$.

After the transition at $P_2$ $q=STABLE$. Choose $\lambda_s$ such that the generated trajectory starting from this point ($P_2$) undergoes its next transition at $P_3 = (s, -n)$. After the transition at $P_3$ we have $q=UNSTABLE$. Since the system dynamics are the same now as they were during
the trajectory from $P_1$ to $P_2$ but with initial conditions of the opposite sign we know that the system will undergo transition to $q = STABLE$ at $P_4 = (-s, -n)$.

The transition at $P_4$ reverts the system dynamics back to the dynamics seen during the trajectory from $P_2$ to $P_3$ so we know that the trajectory will undergo transition to $q = UNSTABLE$ at $P_5 = P_1$.

Since $m$ and $n$ are arbitrary, the eigenvalues are themselves arbitrary. Thus there exists a nominal trajectory for arbitrary $s$, arbitrary $\lambda_u$, and arbitrary $\lambda_x$. 

![Figure 2-2 - Phase portrait of nominal trajectory](image)

*Theorem 2-2* – For the hybrid system defined in accordance with the system description the solution contains a stable limit cycle.

*Proof:* We will show that a closed trajectory generated by this hybrid system is a stable limit cycle by showing that it is the result of an iterated function system with a stable fixed point. Establishing the presence of this stable fixed point will rely on defining a pair of iterated
functions \( f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) that map \( x_2 \) values along a switching surface at the \( i^{th} \) transition time \( t_i \) to \( x_2 \) values along a switching surface at the next transition time \( t_{i+1} \).

The function \( f(m) \) considers trajectories that progress from transition \( e_1 = (STABLE, UNSTABLE) \) to transition \( e_2 = (UNSTABLE, STABLE) \), as seen in Figure 2-3 - Representative phase portrait of trajectory defining \( f(m) \). Specifically, the function \( f(m) \) maps the \( x_2 \) magnitude at \( e_1 \) to that at \( e_2 \).

![Figure 2-3 - Representative phase portrait of trajectory defining \( f(m) \)](image)

The function \( g(m) \) considers trajectories that progress from transition \( e_2 = (UNSTABLE, STABLE) \) to transition \( e_1 = (STABLE, UNSTABLE) \), as seen in Figure 2-4. Specifically, the function \( g(m) \) maps the \( x_2 \) magnitude at \( e_1 \) to that at \( e_2 \).
To show that the system has a stable fixed point we will show that it meets the requirements of the contraction mapping principle (Altman 1983). This will be accomplished by showing that there exists a point $c$ such that $h(c) = c$ and that there exists a region $\mathbb{M}, c \in \mathbb{M}$ where $h'(\mathbb{M}) < 1$ (Kirk and Khamsi 2001).

Proving the existence of $c$ will involve showing that the graph of $h(m)$ intersects the 45 degree line. To aid in visualizing this concept, graphs of representative functions and their composition can be seen in Figure 2-5. In this visualization, the point $c$ lies at the intersection of the 45 degree line (solid) with the curve $h$ (dash-dot).

The existence of this intersection $c$ can be proven by showing that there exists a value $u$ such that $h(u) > u$ (above the 45 degree line) and a value $v$ such that $h(v) < v$ (below the 45 degree line) with $u < v$. Then, by continuity and the intermediate value theorem this will show that there exists $c \in (u, v) \mid h(c) = c$. 

Figure 2-4 - Representative phase portrait of trajectory defining $g(m)$
First, consider the point above the 45 degree line, for which we must show $h(m) > m$. For $m$ we will use the value 0 and show that $h(0) > 0$, by showing that $f(0) > 0$ and that $g(0) = 0$.

To show that $f(0) > 0$, start by considering the trajectory that the subsystem follows from initial point $(-s, 0)$. Since the subsystem that is active in this region is unstable the stability exponent is greater than 0 (see Figure 2-3). Thus the unstable trajectory starting at $(-s, 0)$ will next cross the $x_1$ axis at an $x_1$ value greater than $s$. Since the trajectory is continuous this means that it crosses the line $x_1 = s$ at some positive value. This is $f(0)$ by definition and so $f(0) > 0$.

In order to construct $h(m) = f(g(m))$ next consider the function $g(m)$ by analyzing the trajectory starting on the switching surface at $(s, m)$. The value $g(m)$ can be bracketed between two cases: the marginally stable case and the case of infinitely large damping. Invoking the logic in Theorem 2-1, the bracket on $x_2$ magnitude at transition is between 0 (infinite damping case) and the starting $x_2$ value (marginally stable case). For $m = 0$, the starting $x_2$ value is zero, hence
the bracket is upper and lower bounded by zero. Thus, the transition must occur at \( x_2 = 0 \) and so \( g(0) = 0 \).

Now construct the composition \( h(m) = f(g(m)) \) for \( m = 0 \). Since \( f(0) > 0 \) and \( g(0) = 0, h(0) > 0 \). This means there exists \( u \) such that \( h(u) > u \).

The next step in establishing that the composition \( h \) crosses the equilibrium line is to show that \( h(v) < v \) at some point \( v \). We will prove the existence of \( v \) by showing that, for sufficiently large values of \( m, f'(m) = 1 \) and \( g'(m) < 1 \) and thus \( h'(m) < 1 \) (more specifically \( \lim_{m \to \infty} h'(m) < 1 \)).

Note that the time a trajectory takes to travel between two \( x_1 \) values (\( u \) and \( v \)) is inversely related to the \( x_2 \) value of the trajectory.

\[
\Delta t = \int_u^v \frac{1}{x_2} \, dx_1
\]

To show that \( \lim_{m \to \infty} h'(m) < 1 \) consider the following. First we show for \( m \gg s \) that \( f'(m) \) approaches 1. To see this result observe that, per (2-8), the time spent in the unstable region is inversely related to the distance of the trajectory from the \( x_1 \) axis. Taking the unstable version of (2-6) as a parametric equation then finding the derivative yields:

\[
\frac{dx_2(t)}{dx_1(t)} = \frac{dx_2(t)}{dt} \frac{dx_1(t)}{dt} = \frac{ax_1 + bx_2}{x_2}
\]

for some constants \( a \) and \( b \). Equation (2-9) is finite for \( x_2 \neq 0 \). Consider a trajectory that enters the unstable region at the point \((x_1, x_2) = \pm(s, m)\). Substituting these values \((x_1, x_2)\) into (2-9) and taking the limit for \( m \gg s \), (2-9) becomes
\[
\frac{dx_2}{dx_1} \approx b \tag{2-10}
\]

The slope \(f'(m)\) can be obtained from this equation by invoking the Euler’s method approximation of the trajectory over the unstable region (between \(P_1\) and \(P_2\) or between \(P_3\) and \(P_4\)). Define the \(x_1\) distance traveled over this region to be \(2s\). By the Euler method, the corresponding change in the \(x_2\) coordinate is then given by:

\[
f(m) = m + \frac{dx_2}{dx_1} 2s + \mathcal{O}(4s^2) \tag{2-11}
\]

Since we are considering the case where \(m \gg s\), (2-11) can be further reduced by substituting (2-10). Per (2-8) as \(m\) increases the time spent in this region becomes smaller meaning that the derivative approaches constant over the entire trajectory, and thus the error term disappears.

\[
f(m) = m + 2sb \tag{2-12}
\]

Consider two points \((m_1, m_2)\) on the switching surface. Subtracting \(f(m_2)\) from \(f(m_1)\) and rearranging gives:

\[
f(m_2) - f(m_1) = m_2 - m_1 \tag{2-13}
\]

Note that

\[
f'(m) = \lim_{{m_2 \to m_1}} \frac{f(m_2) - f(m_1)}{m_2 - m_1}. \tag{2-14}
\]

In other words, the limit as \(m_2 \to m_1\) is the derivative of \(f(m)\). Thus, \(f'(m) = 1\) for large \(m\).
To construct a bound for $h'(m)$, the above limiting case for $f'(m)$ must be combined with a bound for $g'(m)$. We will now show for large values of $m$ that $g'(m) < 1$. Since $q = STABLE$ in this region we know that $g(m) < m$. Express this as a ratio.

$$g(m) = rm, \quad r < 1 \quad (2-15)$$

It may appear at first that we cannot guarantee that $r$ will be less than 1. However, for large values of $m$ the system behavior can be analyzed as through it is traveling through the half-plane, not just the half-plane outside of the switching surface. Regardless of how the system might be rotated a stable system will always cross the negative $x_2$ axis at a fractional value of its last $x_2$ axis crossing. Using a similar solution to the ODE found in (2-6) gives a decomposition of $g(m)$ in the desired format.

$$g(m) = me^{\alpha s t} \left( 1 + \frac{\alpha s}{m} \right) \cos \beta t + \left( \frac{s \beta}{m} + \frac{\alpha}{\beta} \right) \sin \beta t$$

Since $m \gg s$ (2-16) can be simplified by ignoring negligible terms.

$$g(m) = me^{\alpha s t} \left( \cos \beta t + \frac{\alpha}{\beta} \sin \beta t \right) = mr \quad (2-17)$$

To show that $g'(m) < 1$ for large values of $m$ observe that taking a standard derivative of (2-15) gives

$$g'(m) = \frac{dr}{dm}m + r \quad (2-18)$$

As can be seen in (2-17) $r$ is not a function of $m$. This might not be obvious at first but in extreme cases (such as those being explored) this is true. The time that it takes for a trajectory to travel through half a period is only a function of $\beta$ so $t$ is not a function of $m$ as might be initially suspected. Far away from the $x_1$ axis the difference in transit time through the right (or left) half
plane approaches the same as the time (2-8) that the trajectory takes through the right (or left) half plane right (left) of the switching surface. Thus,

\[ g'(m) = e^{\alpha t} \left( \cos \beta t + \frac{\alpha}{\beta} \sin \beta t \right) = r < 1 \]  

(2-19)

For large values of \( m \) the trajectory in the stable region approximates a half cycle so \( t \) approaches \( \frac{\pi}{\beta} \).

The derivative of the composition function is \( h'(m) = f'(g(m))g'(m) \). Solutions to systems of linear ODEs are unique (Tenenbaum and Pollard 1985), meaning that trajectories do not cross and thus \( f(m), g(m), \) and \( h(m) \) are monotonically increasing. To see this observe that if they are not monotonically increasing then there exist two values of \( m \) that the function maps to the same value. This would imply that the trajectories intersect at the point defined by the switching surface and the value of the function, a contradiction. As \( m \) goes to infinity, \( g(m) \) also goes to infinity. Hence, according to (2-14), \( \lim_{m \to \infty} f'(g(m)) = \lim_{m \to \infty} f'(m) = 1 \). Then, since \( g'(m) < 1 \), by (2-19), we know \( h'(m) = f'(g(m))g'(m) < 1 \). With the derivative \( h'(m) < 1 \) shown for large \( m \) we know that there is some large value \( m \) where \( h(m) < m \). Choose such a value of \( m \) and call it \( v \). Hence \( h(v) < v \).

Since \( u > h(u) \) and \( v < h(v) \) for \( u < v \), we know by continuity that \( h(m) \) must cross the equilibrium line. In other words there is a \( c \in (u, v) \) such that \( h(c) = c \).

The penultimate step in the proof is to show that there is a region surrounding \( c \) where \( h'(m) < 1 \). Define \( \epsilon > 0 \) to be a small perturbation from \( c \). Assume \( h'(c) \geq 1 \). By definition of the derivative this means \( \lim_{\epsilon \to 0} (h(c) - h(c - \epsilon))/\epsilon > 1 \) however this means \( h(c) - h(c - \epsilon) > \epsilon \) but by continuity the neighborhood of \( c \in \mathbb{E} \) \( \forall \epsilon \in \mathbb{E} \) \( h(c) - h(c - \epsilon) < \epsilon \). This is a
contradiction, thus $h'(m) < 1$ for some region of values $< c$. For values greater than $c$ we have already established that $h'(m) < 1$.

Finally, to show that the region of convergence is a metric space over which $h(m)$ is a contractive mapping, define $\mathbb{M} \subset \mathbb{R}^+$ as the largest continuous region surrounding $c$ such that $h'(m) < 1$ for $m \in \mathbb{M}$. Choose $L = \max_{m \in \mathbb{M}} h'(m)$. Since $h'(m) < 1$ we know that $L < 1$. Since $h(m)$ is continuous this means that $\forall m, n \in \mathbb{M}: |h(m) - h(n)| \leq L|m - n|$. In other words, the mapping $h$ is contractive.

This means that the point $h(c) = c$ meets all of the prerequisites for the contraction mapping principle and as such the solution to this iterated functional system has at least one stable fixed point. This fixed point represents a point that is recurring at every half iteration and further implies that the system has a limit cycle. Since the fixed point is attractive the limit cycle is stable.

2.5.2 Specification of Bulk Properties

The purpose of this section is to provide the designer with the tools needed to find the gains and switching surface value needed to control a system with desired properties. Additionally these theorems show how certain properties are tunable to allow an operator to vary them. An iterative approach may need to be employed in order to arrive at a solution that meets all of the specifications. However, the Design Procedure section will show how the number of iterations can be limited.

*Theorem 2-3:* For a given set of $K_s, K_u$ increasing the value of $s$ increases the amplitude of the limit cycle.
Proof: From (2-12) it can be seen that increasing s increases the value of $x_2$ at the next switching surface. By uniqueness of solution to set of linear equations this means that in the STABLE region the trajectory for the system with the larger value of s will not cross the trajectory of system with the smaller value of s. Thus, when the system with the larger value of s crosses the $x_1$ axis it will do so at a larger value of $x_1$ than the system with the smaller s. By definition this means that the amplitude is larger. ■

Theorem 2-4: increasing $\beta_s$ or $\beta_u$ increases the frequency of the associated limit cycle.

Proof: This is directly evident from observing the solution given in (2-6) (and the corresponding stable version). Since the period of the subsystem is inversely proportional to its associated $\beta$ changing that $\beta$ changes both the subsystem frequency and the overall frequency of the hybrid system. To see that change in frequency caused by changing $\beta$ is not dependent on what part of the cycle the subsystem is in observe that changing $\beta$ is a simple compression of the sine wave meaning that the time it takes to get between any two function values is varied linearly with $\beta$. ■

2.6 Design Procedure

Finding the switching surface value and the gains needed to achieve the desired limit cycle follows a three step procedure.

- Find suitable starting values based on the model of the system to be controlled
- Apply theorem 2-4 to find $\beta$ values which give the desired frequency
- Apply theorem 2-3 to find the switching surface value which gives the desired amplitude

With a desired amplitude and frequency in mind and a good model of the system the designer starts with initial stable and unstable eigenvalues near the eigenvalues of the
uncontrolled system. This gives the $\alpha$ that will be used throughout the design procedure. The hybrid system is then simulated to find the emergent frequency and amplitude. Application of Theorem 2-4 allows for the frequency to be adjusted. It has been found that adjusting $\beta$ for either subsystem affects the amplitude to some extent. As to whether $\beta_s$ or $\beta_u$ should be adjusted for the best result will require further investigation. Once the desired amplitude has been achieved focus turns to Theorem 2-3 which is used to set the amplitude. It does not appear that changing the location of the switching surface changes the frequency of the limit cycle. In the theoretical case of a perfectly circular limit cycle it can be shown analytically that the frequency is constant. It appears that a similar phenomenon exists for non-circular limit cycles.

By the application of these two tuning methods the designer can thus construct a set of gains and a position for the switching surface such that the hybrid system produces a stable limit cycle of the desired amplitude and frequency.

How much to adjust the parameters is still very much ad hoc. However, Theorem 2-3 and 2-4 suggest a simple line search as this is guaranteed to converge to a solution. The implementation used in the thesis section adjusted the parameters using a weighted adjustment based on how far off the iteration was from having the desired properties.

2.7 Example Case

Suppose a designer is tasked with creating a controller which produces a stable limit cycle trajectory of amplitude 0.25 and frequency of 0.5 Hz when applied to a base system with state matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & -10 \end{bmatrix}$ and input matrix $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
As a starting point, let’s select $s = 0.2$, $\lambda_s = -0.1 \pm 0.4i$, and $\lambda_u = 0.1 \pm 0.4i$. Performing a numerical simulation of this shows that the amplitude is 1.13 and the frequency is 0.019 Hz. Applying the Design Procedure gives a configuration that matches the desired properties.

<table>
<thead>
<tr>
<th>Table 2-1 - Design values and Bulk Properties for Example Configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>Value</td>
</tr>
</tbody>
</table>

If it is then desired to raise the amplitude to 0.5 altering $s$ to a value of 0.2121 gives an amplitude of 0.5000, without changing the system frequency.

2.8 Conclusion

Limit cycles are a highly desirable property for systems to have since it permits the system to reject disturbances without active tracking by the controller. By applying a hybrid controller which switches between stabilizing and destabilizing the system based upon the nearness of the system trajectory to the origin in one variable, a linear second order system can be forced into a limit cycle.

The amplitude and frequency of the limit cycle is a product of what gains are chosen to stabilize/destabilize the system and how close the system trajectory has to be to the origin for the control to switch modes. These readily tunable properties allow for designer specification of stable limit cycles and create an understandable context for the operator to manipulate the cycle’s properties during operation.
3 Template for Robust Soft-Body Crawling with Reflex-Triggered Gripping

3.1 Introduction

The central premise of this thesis chapter is to advance the hypothesis that underactuated soft-bodied systems can produce robust gaits using limited sensory feedback to control interactions with the surrounding environment (or *substrate*). For a soft body system consisting of connected segments, securing the soft-body system to the substrate limits the active length of the body segments and provides structure for locomotion (e.g., known positions through which the curve of the body must pass).

This study is motivated by the remarkable ability of caterpillars to traverse difficult terrain, including tight spaces, vertical stalks, and compliant surfaces. This ability is made even more remarkable because it functions through many stages of the caterpillar life cycle, even as their bodies undergo drastic changes in size (van Griethuijsen and Trimmer, Locomotion in Caterpillars 2014). For example, caterpillars of the model species *Manduca sexta*, commonly known as the tobacco hornworm, grow 10,000 fold in mass before metamorphosis (Lin, et al. 2010). That they appear to use the same neural circuitry to traverse all types of terrain at all different size scales (van Griethuijsen and Trimmer, Locomotion in Caterpillars 2014) suggests that caterpillars are a living solution to a truly challenging, unsolved problem in engineering control: that of developing a robust controller to enable movement over varied terrain in the absence of a detailed model of the system being controlled.

One of the most salient features of caterpillar crawling is their use of tiny barb-shaped grippers, called crochets, that are situated at the tip of the caterpillar’s main appendages, the
abdominal prolegs (Barbier 1985). Crochets attach passively to nearly any substrate in contact with the prolegs, such that caterpillars need little energy to hang vertically or upside down (Belanger and Trimmer 2000). Because crochets attach passively, however, locomotion can only occur if muscles disengage crochets, releasing the substrate and allowing the associated proleg to take a step (Belanger and Trimmer 2000). It is very clear that crochets play a key role in caterpillar mobility, contributing greatly to the caterpillar’s ability to climb even sheer vertical surfaces (van Griethuijsen and Trimmer 2009). Crochets have recently been postulated to play a more subtle role in locomotion, as well, attaching the caterpillar such that a rigid substrate serves as an external skeleton for the caterpillar’s soft body (Lin and Trimmer 2010). Some prior investigations into duplicating caterpillar-like locomotion have not accounted for the role that the crochets play (Orki, et al. 2012), however their physiological importance indicates that they must be considered. Although movements of the prolegs have been studied extensively (Mezoff, et al. 2004) (Trimmer and Issberner 2007) it is unclear how crochet release is synchronized with the anterograde wave of body muscle activation driving locomotion.

Computation provides one means of studying crochet synchronization. In this approach, computers solve a concise set of mathematical equations, which is sometimes called a template (Full and Koditschek, Templates and Anchors: Neuromechanical Hypotheses of Legged Locomotion on Land 1999). The template captures the essential locomotion characteristics of an animal. In the case of caterpillar crawling, some mathematical models already exist (Saunders, Golden, et al. 2011), (Saunders, Trimmer and Rife, Modeling locomotion of a soft-bodied arthropod using inverse dynamics 2011). Existing mathematical models do not explicitly consider substrate gripping, however, so they cannot serve as templates for studying the
synchronization of proleg attach and release. New mathematical models are needed for this purpose.

Templates are powerful not only in comparative biology, but also in engineering, where they can be used for synthesis of new robotic systems. As an example, the Spring-Loaded Inverted Pendulum (SLIP) was introduced as a model to describe hopping and running in bipeds (Blickhan 1989). The model was later recognized as a template describing walking and running in a wide range of animal species including quadrupeds and hexapods (Full and Tu, Mechanics of a Rapid Running Insect: Two-, Four-, and Six-legged Locomotion 1991). Eventually this template was applied to develop a wide range of engineering technologies from pedestrian navigation systems (Liu, Osechas and Rife 2012) to highly mobile hexapod robots capable of traversing uneven terrain (Spenko, et al. 2008). We anticipate that a template for soft-bodied crawling could likewise serve both as a tool for understanding natural locomotion and for engineering locomotion in synthetic, soft-bodied robots.

In constructing a template for soft-bodied crawling, different approaches might be pursued to control the timing of gripper attachment and release. Two reasonable biologically-based options are a global central pattern generator (CPG) and a local reflex-based controller. CPGs are rhythmic waves of motor activity that appear in many animal species (Ghigliazza and Holmes 2004), (Guertin 2009) and that have also been emulated in robots (Ijspeert 2008) (Ayers and Witting 2006). In M. sexta rhythmic waves of motor activity can be produced by pharmacologically stimulating the isolated nerve cord (Johnson and Levine 1996); however, in the absence of normal sensory feedback, these waves are different from the motor patterns recorded in crawling caterpillars. During natural behavior sensory feedback interacts dynamically with the neural activity produced by CPGs and this can dominate the resulting
motor output. Sensory stimuli can also trigger local motor responses to produce reflexive movements. In *M. sexta* there is a strong proleg withdrawl reflex produced by deflecting sensory hairs close to the crochets (Trimmer and Weeks 1991) (Weeks 1987) but this is context dependent and it is used as an assistance reflex to avoid physical obstacles during crawling. In short, experimental evidence shows that both CPG and reflex control exist in caterpillars but it is unclear how these might be integrated to control proleg gripping.

Given that current experimental work has not yet identified the physiological mechanisms governing gripper synchronization, this thesis chapter focuses on a reflex-based model for gripping for several reasons. First, gripping can be evoked by stimulating medial and ventromedial sensory hairs on the abdomen (Mezoff, et al. 2004) and CPGs cannot account for adaptive motion observed in real animals (Pearson 1987) (Stevenson and Kutsch 1987). Second, reflex-based control provides the desirable property that it is highly fault tolerant, an advantage also leveraged by engineers in constructing localized sensory-feedback mechanisms, which they call *decentralized controllers* (Bakule n.d.). Third, reflex-based control has previously been demonstrated to be effective in other efforts to use computers to simulate animal crawling, and in particular to provide high fidelity prediction of kinematic data for linked segment crawlers, like snakes (Chen, Ma and et. al. 2007) (Date and Takita 2007).

The remainder of this thesis chapter develops the template for soft-body crawling and analyzes its robustness to parametric uncertainty. The chapter is organized as follows. The template is constructed in two parts. First the basic physics model is introduced: that of a flexible body consisting of a series of forced spring-mass-damper elements. Subsequently, the template is completed by proposing reflex-based control rules to modulate muscle activation and gripper attachment/release. After introducing the template, the chapter presents a computational
sensitivity analysis. The sensitivity analysis details the degree to which simulated crawling changes in response to large perturbations in template parameters, such as those describing the external environment (e.g. slope of the substrate) and animal morphology (e.g. segment length, segment mass, etc.). Findings are discussed with an eye toward how the template might be applied to future applications in comparative biology and robotics engineering.

3.2 Physics-Based Template

Investigation into how localized sensing and actuation can coordinate movements throughout an entire system requires the specification of a quantitative motion model and a control law. This section describes a physics-based model (or template) that is intentionally very simple, with just enough complexity to capture the essence of caterpillar motion. Templates reduce the complexity of the actual physical animal (called an anchor to the template) to reveal how the many degrees-of-freedom of the actual system self-organize into a pattern that enables locomotion. Whereas the proposed template is only a six degree-of-freedom dynamic model, real caterpillars have hundreds of muscles that are coordinated in a three dimensional architecture.

The core idea of the proposed template is to recreate a stereotypical caterpillar crawling pattern, one that can be described as a traveling anterograde wave (i.e., a wave moving from the tail toward the head) (Simon, Fusillo, et al. 2010). The crawl is initiated when the gut is thrust forward, carrying with it the rearmost caterpillar legs in what is called visceral-locomotory pistoning (Simon, Woods, et al. 2010). Muscles then carry the rest of the body segments forward ultimately resulting in the forward locomotion of the animal. Perhaps the most critical element controlling forward locomotion is the attachment and detachment of the crochets synchronous with the anterograde wave of muscle activation (Belanger, 2000 #3209).
To recreate the anterograde wave, we have adapted a modeling approach previously used to analyze rigid-skeleton snake locomotion, using curvature based control laws (Takahide, et al. 2012). The physics-based snake models featured two-dimensional rigid links connected by springs and dampers. For the analysis of soft-body systems presented here, rigid links have been replaced by compressible sections. In effect the proposed soft-body crawling model is one-dimensional, capturing deformations along the anterior-posterior axis. Deformations in the dorsal-ventral direction can be approximated, indirectly, using simple geometric relations.

3.2.1 Fundamental Concepts

The proposed soft-body crawler template is a physics-based model that consists of a one-dimensional sequence of masses joined by springs, dampers, and actuators. These elements represent reduced-order models for biological tissue, including muscle and cuticle (Lin, et al. 2010). Modeling of the forces generated by the springs and dampers is done using conventional linear models (Meriam and Kraige 2009). Each segment also has a crochet which can be activated to attach the caterpillar to the substrate. Body segments are treated as lumped masses (Saunders, Golden, et al. 2011), (Saunders, Trimmer and Rife, Modeling locomotion of a soft-bodied arthropod using inverse dynamics 2011). The thoracic portion of the caterpillar is lumped into a single segment that is connected to the rest of the segments by a spring and damper only. Since the segments are treated as point masses the length of a segment is taken as the distance between a given mass and next most posterior mass.
Figure 3-1. Overview of the model showing the lumped mass body segments connected to each other.

Figure 3-2. A single body segment schematic showing the spring, damper, and muscle attachments

Position and velocity of the $i^{th}$ lumped mass is given in the global coordinate system by $p_i$ and $v_i$, respectively. Length of a segment ($l_i$) is then given by the absolute distance between adjacent lumped masses. Since spring force is a function of the displacement from some equilibrium point a rest length $\bar{l}_i$ is introduced. Mathematically the spring and damper forces are:

\[ Spring \ force: \ F_s^i = -k_i(l_i - \bar{l}_i) = -k_i(p_i - p_{i-1} - \bar{l}_i) \]  \hspace{1cm} (3-1)

\[ Damper \ force: \ F_d^i = c_i v_i = -c_i \frac{d}{dt} (l_i) = -c_i \dot{l}_i \]  \hspace{1cm} (3-2)

Here $k$ is the spring stiffness, $c$ is damping coefficient, and $F_m$ is a scalar describing the magnitude of force generated by an active muscle. An additional forcing term can be added to
each segment to explore the effects of gravity on the system. The gravitational force per segment is modeled as \( m_l g \sin(\theta) \), where \( m_l \) is the segment mass, \( g \) is the gravitational acceleration, and \( \theta \) is the orientation of the caterpillar relative to the horizontal plane (where \( \theta \) equal zero implies horizontal crawling).

In addition to the linear components of the model there are two non-linear elements essential to locomotion: the muscles and crochet models. The muscle model has only two states: ON and OFF, and the corresponding tensile force is held constant as long as the muscle is active. Muscle activation in a given segment is described by a binary variable \( \phi_i \).

\[
\text{Muscle force: } F^i_M = \begin{cases} 
0 & \phi_i = 0 \\
A_l & \phi_i = 1 
\end{cases}
\]  

Similarly the crochets are a binary system in which they are either ATTACHED or FREE with this state stored in the discrete variable \( \psi_l \). When the crochets are attached they do not move and the corresponding body segment is pinned to the substrate. In the proposed, model the thoracic segment does not have crochets or a muscle attachment to A3. During crawling the
periodic actions of the crochets effectively reduce the animal’s degrees of freedom. This allows the contraction of muscles to simply shift the mass of the caterpillar body in the forward direction.

It is important to note that while, in general, spring-mass-damper systems may exhibit oscillations and resonance, caterpillars do not. This apparent conundrum is resolved by introducing high damping. High damping removes kinetic energy quickly, so that inertial forces remain small (implied by the known kinematics of *M. sexta* (Trimmer and Issberner 2007)) and oscillations do not occur.

### 3.2.2 Coupled Differential Equations

Consider a collection of lumped masses linked together in a one-dimensional chain of length $N$. In the caterpillar-inspired case illustrated in Figure 1, $N$ has been chosen to be 6, accounting for five distinct body segments with prolegs and one thorax. Except for the initial and terminal lumped masses each interior mass has two indexed sets of forces described by (3-1) (3-3). That is, the $i^{th}$ mass is accelerated by muscle, damper, and spring forces from both an anterior segment ($i$) and a posterior segment ($i+1$). Additionally there is a projected gravitational force acting along the caterpillar’s direction of motion and a crochet force. This gives rise to a system of 2nd order differential equations. The differential equation for each lumped mass is:

$$m_i \ddot{q}_i = (F^i_M - F^{i+1}_M) + (F^i_d - F^{i+1}_d) + (F^i_s - F^{i+1}_s) + \kappa_i - m_i g \sin(\theta)$$  \hspace{1cm} (3-4)

In order to interpret this set of dynamic equations, it is useful to reduce the number of parameters using an appropriate nondimensionalization. In the process of nondimensionalizing, we also simplify the model by assuming the damping coefficient, stiffness, muscle force, segment reference length, and segment mass are identical for all segments. Applying this
assumption, substituting (3-1) (3-3) into (3-4), and then dividing by mass \( m \) gives the following differential equation.

\[
\ddot{p}_l = -\frac{c}{m} \nabla^2 p_l - \frac{k}{m} \nabla^2 p_l + \frac{A}{m} (\phi_l - \phi_{l+1}) + \frac{\kappa_l}{m} - g \sin(\theta)
\]  

(3-5)

The lumped-mass positions \( p_l \) are excited by two control inputs: muscle activity (via \( \phi_l \)) and crochet activity (via \( \kappa_l \)). The Laplacian operator \( (\nabla^2) \) is used as a shorthand for a spatial second-difference, which might loosely be referred to as a curvature.

\[
\nabla^2 p_l = -p_{l+1} + 2p_l - p_{l-1}
\]  

(3-6)

\[
\nabla^2 \dot{p}_l = -\dot{p}_{l+1} + 2\dot{p}_l - \dot{p}_{l-1}
\]

The uniform reference lengths \( \bar{l}_l \) from (3-1) cancel in forming the Laplacian. Note, the sign convention typically used for the Laplacian operator is opposite that used here. This is due to the direction of positive motion being defined in the opposite of increasing segment number. (Numbering of the segments in this manner was selected to correspond with the biological numbering convention.)

To complete the nondimensionalization of equation (3-5) requires the introduction of a nondimensional length \( \chi \) and time \( \tau \).

\[
\chi = \frac{pk}{A}
\]  

(3-7)

\[
\tau = t \sqrt{\frac{k}{m}}
\]  

(3-8)

Substituting these definitions in (3-5) gives the following.

\[
\ddot{x}_l = -2\zeta \nabla^2 \dot{x}_l - \nabla^2 \chi_l + (\phi_l - \phi_{l+1}) - G - K
\]  

(3-9)
Note that the muscle activation parameters $\phi_i$ are inherently dimensionless (one when on, zero when off). Thus, the original set of six dimensional system parameters ($m, c, k, A, g, \kappa$) is reduced to a smaller set of three nondimensional parameters: the damping ratio $\zeta$, the dimensionless gravity $G$, and the dimensionless crochet force $K$.

$$\zeta = \frac{c}{2\sqrt{mk}}$$

$$G = \frac{mg\sin(\theta)}{A}$$

$$K = \frac{\kappa}{A}$$

Further simplification is possible if the crochet is modeled as a gripper that halts acceleration of a segment (when attached) but that produces no force otherwise (when free). To distinguish these two cases, a binary crochet control variable $\psi_i \in \{FREE, ATTACHED\}$ is introduced. The resulting hybrid dynamic system retains only two of the nondimensional parameters from (3-9), $\zeta$ and $G$.

$$\psi_i = 'FREE': \quad \ddot{x}_i = -2\zeta \nabla^2 \dot{x}_i - \nabla^2 \dot{x}_i + (\phi_i - \phi_{i+1}) - G$$

$$\psi_i = 'ATTACHED': \quad \ddot{x}_i = 0, \dot{x}_i = 0$$

This model fully describes the dynamics of the segment positions ($\chi_i$) as a function of the control variable inputs ($\phi, \psi$), when boundary conditions are defined for the first and last segments ($\chi_0$ and $\chi_N$) and when initial conditions are defined for all segments. The boundary condition at the terminal proleg initiates a crawl, defined as a discrete wave of motion that begins and ends with all crochets attached to the substrate. To capture this effect as simply as
possible, the terminal proleg position \( \chi_N \) is specified as a step function proportional to the rest length of a segment. Here the coefficient of proportionality is given the label \( \chi_{\text{init}} \).

\[
\chi_N = \begin{cases} 
0 & t = 0^- \\
\chi_{\text{init}} \cdot \bar{I} & t \geq 0^+ 
\end{cases}
\]  

(3-15)

Boundary conditions for the most anterior segment (that is, the thoracic lumped mass) were constructed with the assumption that the thoracic legs, unlike the prolegs, are not able to fully stop the motion of the segment. Additionally the muscle connecting the thoracic lumped mass and the A3 lumped mass is assumed to not play a role in the motion. This is enforced by explicitly setting \( \psi_0 = \text{'FREE'} \) and \( \phi_0 = 0 \). Here again the rest length enters the model because the spring forces on the A3 segment are unbalanced so the rest length does not cancel from the Laplacian operator for this specific segment.

The initial conditions for segment velocities are set to zero. Initial conditions for segment positions are set to ensure a periodic steady-state condition. In other words, the initial positions of each segment, relative to the terminal proleg position, are set to be the same both at the crawl’s start (time 0) and end (time \( \tau^+ \)).

\[
\chi_i(\tau^+) = \chi_i(0) + \chi_N(\tau^+) - \chi_N(0)
\]

(3-16)

Although it is difficult to compute \textit{a priori} what the initial condition (3-16) should be to achieve a periodic steady-state, initial conditions can found in practice by iteratively simulating crawls until all transients decay and only the periodic steady-state condition remains.
3.3 Control Method

In order for the proposed physics-based model to generate anterograde waves, control rules for its two actuators must be specified. Specifically, the two actuators are the muscle and crochet inputs, \( \phi \) and \( \psi \).

As mentioned in the introduction, a reflex-based methodology was used to assign mathematical models for both types of actuation. Notionally, these control laws reflect feedback on two types of physiological sensor present in \( M. sexta \). The first physiological sensor is the segmental stretch receptor. Segmental stretch receptors are poorly suited for providing feedback about absolute displacements (Simon and Trimmer 2009); however, these sensors could, in principle, provide local information about changes in length between neighboring segments. The second physiological sensors are the filiform hairs (planta hairs) at the tip of each proleg. Planta hairs act as proximity sensors that could, in principle, detect when the proleg is close enough and moving slow enough to attach to the substrate (van Griethuijsen and Trimmer, Caterpillar crawling over irregular terrain: anticipation and local sensing 2010).

The proposed muscle-activation control law reacts to differences in length (stretch) between sequential segments. In other words, control is related to the Laplacian operator.

\[
\nabla^2 \chi_i < 0 \Rightarrow \phi_i = 1 \quad (3-17)
\]

\[
\nabla^2 \chi_i > 0 \Rightarrow \phi_i = 0 \quad (3-18)
\]

Two-dimensional curvature-based control has previously been employed in snake-like robot modeling (Date and Takita 2007), and it is natural to extend the concept to the one-
dimensional model presented here, noting the close mathematical relationship between the Laplacian operator and curvature.

By contrast, since gripping is a unique feature of the soft-body crawling template, it is difficult to identify prior research that can guide the formation of an appropriate control law. In this light, a new control law is proposed. The controller acts differently whether the gripper is \textit{ATTACHED} or \textit{FREE}.

When \textit{ATTACHED}, the control law triggers gripper release only if the sum of forces on the associated segment becomes sufficiently high. In other words, differential muscle activation leads to release. Mathematically, the release condition is defined using a threshold $A_{\text{acc}}$ that defines the transition from high to low.

$$\nabla^2 \chi_i + (\phi_i - \phi_{i+1}) - \mathcal{G} > A_{\text{acc}} \Rightarrow \psi_i = '\text{FREE}'$$

When \textit{FREE}, attachment of the gripper occurs when three conditions occur at the same time for a given segment.

- The resultant of forces on that segment (corresponding to its acceleration) must be low (below $A_{\text{acc}}$)
- The speed of the segment must be low (below $A_{\text{vel}}$), as the crochets may not able to engage if the segment is moving too fast.
- The length of the segment must be a large fraction of its rest length (above $A_{\text{pos}}$), a one-dimensional proxy, as described below, which represents the condition that the proleg is in contact with the substrate.

Together, the attachment gating conditions are:
\[ \ddot{x}_t < \Lambda_{acc}, |\dot{x}_t| < \Lambda_{vel}, \chi_{t+1} - \chi_{t-1} > \Lambda_{pos} \Rightarrow \psi_t = 'ATTACHED' \] (3-20)

To better understand the sense in which the last (positional) condition is a surrogate for the proximity of the proleg to the substrate, consider the diagram shown in Figure 3-4. The figure shows three successive segment masses, where the outer segments are attached to the substrate and the middle free. If the segments were rigid (with lengths set to the segment rest length \( \bar{l} \)), then the height of the free segment could be computed geometrically. This height of the free segment (corresponding to the displacement of the segment away from the substrate) must be low in order for crochets to attach, and hence the distance spanned by the three segments ought to approach \( 2\bar{l} \) as a condition for attachment. A relaxed form of this attachment rule is leveraged in (3-20), one which allows for attachment even at somewhat shorter distances. The relaxation accounts for curvature and compression of the soft system. For the purposes of analysis, a nominal configuration is defined in which compression is permitted up to 30% (e.g. \( \Lambda_{pos} \) set to \( 1.4\bar{l} \)) without restricting attachment. Other values of \( \Lambda_{pos} \) are also considered in a sensitivity study.

Figure 3-4. Representation of rigid/compressible bodies in relation to triangle inequality.
Together, the two control laws can be formulated as a pair of hybrid systems which take local information to determine their state and switching conditions. These hybrid systems are illustrated in Figure 3-5, below. The muscle activation hybrid system introduces a force to alter the trajectory of the system. The crochet hybrid system directly alters the differential equations that describe the motion of the caterpillar over time. Through the interaction of these two controls the physical model achieves locomotion and is able to reject certain variations and disturbances.

Figure 3-5. Hybrid system description for the crochet (left) and muscle (right) activation.

3.4 Methods

Computer simulations are used to assess the robustness of the proposed soft-body crawling template to parameter uncertainty. In the same manner that biological caterpillars are relatively
Insensitive to uncertainty in the configuration of their bodies or the environment, a successful template should likewise be insensitive to uncertainty in model parameters.

3.4.1 Simulation Success and Failure

To assess sensitivity, a series of simulations were run in which each model parameter was varied individually, holding all other model parameters equal. An exhaustive list of all model parameters was considered. In all, only seven parameters are needed to describe the system dynamics, initial conditions, boundary conditions, and control: $\zeta, G, \chi_{init}, \bar{t}, \Lambda_{acc}, \Lambda_{vel}, \Lambda_{pos}$. The nominal values for these seven nondimensional parameters are listed in Table 1.

Table 3-1 Nominal parameter values for simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\zeta$</th>
<th>$G$</th>
<th>$\chi_{init}$</th>
<th>$\bar{t}$</th>
<th>$\Lambda_{acc}$</th>
<th>$\Lambda_{vel}$</th>
<th>$\Lambda_{pos}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>1.5</td>
<td>0</td>
<td>0.4</td>
<td>1</td>
<td>0.2</td>
<td>0.012</td>
<td>0.7</td>
</tr>
</tbody>
</table>

These nominal parameter values were not selected based on animal data, since it is not immediately clear how to map actual caterpillar motion to the parameters of the reduced-order template presented here. Instead, the nominal parameters were selected by tuning them to produce simulated gait kinematics representative of experimental data, measured for *M. sexta* and previously reported in (Trimmer and Issberner 2007). A subsequent sensitivity analysis provides further justification for the reasonableness of the nominal values reported in the table.

Sensitivity to parameter variations was assessed by increasing and decreasing each parameter from its nominal value (listed in Table 1) while holding all other parameters constant. After incrementally moving each parameter value moving away from nominal by a discrete step, a simulation was run by integrating the template differential equations using numerical methods.
(a modified Runge-Kutta method (Kincade and Cheney 2009) implemented in Matlab). For each simulation, a set of failure conditions was checked. The upper and lower bounds on allowable parameter values were determined by identifying the first parameter value (moving up or down from nominal) that triggered any of the following four failure conditions.

Failures were declared under the following four conditions.

- **Anterograde wave not observed**: The muscle activation sequence was expected to proceed from the terminal proleg forward toward the thorax. Simulations in which muscle activation occurred out of order were classified as failures.

- **Segment order preserved**: Segments were not allowed to cross. In other words, if the unphysical condition occurred in which segment position advanced ahead of its anterior neighbor’s position, then the simulation was declared a failure.

- **Failure to converge to a steady crawl**: The simulation was initialized by running through ten successive crawls. If the change in segment length from the initiation of one crawl to the next did not converge within 10% of the segment length, then the simulation was declared a failure.

- **Failure of all crochets to engage**: A crawl is defined to end when all crochets reattach to the substrate. A failure condition was introduced to prevent crawls of infinite-duration. This was implemented by declaring the simulation a failure if, for any crawl persists for 100 nondimensional time units without all crochets attaching (as compared to the duration of a typical crawl, which lasts only about 10 nondimensional time units).

Since failure was not guaranteed for any individual parameter, a limiting case was also introduced beyond which that parameter was no longer varied. Specifically, if no failure
occurred before the parameter reached ten times its nominal value (increasing) or negative ten times its nominal value (decreasing), then the sensitivity study was terminated and the parameter was declared highly insensitive.

As a check on these single-parameter sensitivity studies, several additional two-parameter sensitivity studies were conducted, in which a pair of parameters were varied independently, holding all other parameters equal.

### 3.4.2 Distance Traveled

Although the primary output of simulations was a binary metric describing the success or failure of crawling for each set of input parameters, a secondary performance metric was also considered: distance traveled by the thorax segment ($p_0$). Over a large number of runs, distance traveled (normalized by segment length) depended on only one system parameter: the terminal proleg’s (TP) stepping distance at the initiation of each crawl. Distance traveled per crawl (normalized by segment length) was proportional to the dimensionless TP stepping distance $\chi_{init}$. This result corresponds to the observed in *M. sexta* (Simon, Fusillo, et al. 2010) (Trimmer and Issberner 2007) and is not unexpected since steady state crawling requires that on average the thorax moves the same distance as the TP.

### 3.5 Results

#### 3.5.1 Qualitative Gait Comparison

To verify the nominal template parameter values, simulations were run and compared directly to experimental data. Two types of observables were compared: (i) timing data distinguishing the swing phase and stance phase for each proleg and (ii) kinematic data describing the height of each proleg above the substrate (as a function of time).
Crochet attachment timing data are very similar between the experiment and simulation, as visualized in Figure 6. The plot illustrates the duration of the swing phase for each of four prolegs (A3-A6). Biological data are plotted as a box which contains the middle two quartiles of the experimentally observed data. The beginning of the box indicates the liftoff of the proleg, and the end of the box indicates touchdown (defined as marker heights falling below 10% of their maximum value). Times are normalized by the duration of one crawl (from release of A6 to the attachment of A3). As such the horizontal axis, labeled Normalized Cycle Period, has a zero value corresponding to the liftoff of A6 and a unity value corresponding to the touchdown of A3. Corresponding liftoff and touchdown times were assessed for the model as the times of crochet activation (switch to ‘FREE’ state) and deactivation (switch to ‘ATTACHED’ state). The duration over which the model crochets were in the ‘FREE’ state was selected as a means of describing the swing phase for the model; the duration that each crochet remains in the ‘FREE’ state is plotted in Figure 3-6 as a black bar.

The phased order of swing for sequential prolegs, and the relative duration of the swing phase for each proleg, are analogous between the model and experiment. The major distinction
between the model and the biological data is that the retrograde wave propagates faster in the model case than in the experiment, in the sense that each successive leg begins its swing sooner (as a fraction of the total swing length) in the model case. As a result, A3 touchdown occurs sooner in the model case, and so the length of the gait period is shorter (and the duration of the swing for each proleg appears longer for the model, as seen in Figure 3-6). Future work will assess whether a better match between model and biological data is possible for a different set of nominal parameters.

Kinematic data, likewise, indicates reasonable similarity between modeled and experimentally observed gait patterns. Experimentally acquired and smoothed kinematic data for proleg height as a function of time are plotted on the left side of Figure 3-7. Inferred height for the simulated segments is plotted on the right side of Figure 3-7. The inferred height was computed using a rigid triangle model for single free segments, as illustrated in Figure 3-4. (Symmetric trapezoids were used for cases with multiple adjacent free segments.)

The peak heights normalized by segment length were twice as high for the simulation as for the experimental data. Because a more refined model for the shape of the soft segments would allow them to compress and curve, it is reasonable to expect that the peak height might well be approximately half that predicted by the triangle model (again, see Figure 3-4). The fact that the results agree within a factor of two is thus a strong verification for the reasonableness of the simulated model.
Figure 3-7. (left) Vertical displacement of prolegs during a crawl (adapted from (Trimmer and Issberner 2007)). (right) Vertical displacement of body segments interpolated from segment length and crochet attachment positions with applied loess smoothing.

### 3.5.2 Sensitivity Results

The parameter value at which a failure first occurred (all other parameters held constant) were recorded as described in the Methods section. These limits are tabulated in Table 2, for both the decreasing direction (column labeled “Min Value to Failure”) and the increasing direction (column labeled “Max Value to Failure”). In two cases ($\zeta$ and $\bar{I}$), no failure was observed even when the nominal parameter value was increased by a factor of ten. These cases are indicated in the table by the “+” notation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Min Value to Failure</th>
<th>Max Value to Failure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>0.3</td>
<td>+</td>
</tr>
<tr>
<td>$\bar{G}$</td>
<td>-0.35</td>
<td>0.13</td>
</tr>
<tr>
<td>$\chi_{init}$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\bar{I}_i$</td>
<td>0.1</td>
<td>+</td>
</tr>
<tr>
<td>$\Lambda_{acc}$</td>
<td>0.15</td>
<td>0.74</td>
</tr>
<tr>
<td>$\Lambda_{vel}$</td>
<td>0.005</td>
<td>0.15</td>
</tr>
<tr>
<td>$\Lambda_{pos}$</td>
<td>0.25</td>
<td>0.95</td>
</tr>
</tbody>
</table>
Studies conducted in which two parameters were varied independently, holding all other parameters equal, largely confirmed the results of the single-parameter sensitivity studies compiled in Table 3-2. As such, only the results of one representative two-parameter study are included here. Specifically, results are included for the case in which damping ratio $\zeta$ and non-dimensional gravity $G$ were varied simultaneously (see Figure 3-8). The figure arranges the result of each simulation on a $\zeta$-$G$ grid, where failure cases are left unmarked and successful cases are indicated with a star.

![Figure 3-8. Representative Dual-Parameter Sensitivity Result](image)

The dual-parameter sensitivity study confirms that the system is very weakly dependent on the damping ratio parameter $\zeta$, as failure occurs commonly above the $G$ value listed in Table 2 ($G = 0.13$) for most of the highly overdamped cases tested ($\zeta > 1.3$). The exception is near critical damping ($\zeta$ approaching unity), where higher levels of gravity can be tolerated,
presumably because, at lower levels of damping, the model’s springs store additional energy that can be exploited to overcome gravity. It should be noted that this region of low damping was explored for completeness. Biologists have yet to discover a living caterpillar specimen that is not highly damped (e.g., with qualitatively assessed damping ratio $\zeta \gg 1$).

3.6 Discussion

The sensitivity studies suggest that the proposed soft-body crawling template is highly robust to variations in internal and external system parameters. These results are by no means proof that caterpillars employ reflex-based gripper control during nominal walking; however, they strongly support that such a mechanism is a practical means of effecting locomotion in the face of uncertainty.

The sensitivity analysis, together with the mathematical modeling described earlier in the chapter, provide significant insight into the simulated system’s robustness. Altogether, it is possible to identify three mechanisms that make the soft-body crawling template insensitive to uncertainty in various parameters. These are mathematical robustness, physical robustness, and biological robustness. The first mechanism, mathematical robustness, reduces the number of parameters to which the system might be sensitive by replacing a large set of dimensional parameters with a smaller set of nondimensional parameters. The second mechanism, physical robustness, recognizes that certain nondimensional parameters have little impact on crawling because of the constraints imposed by the laws of physics. The third mechanism, biological robustness, mitigates uncertainty in all remaining nondimensional parameters by various biological processes, for example, evolution, adaptation, and/or sensory feedback.
3.6.1 Mathematical Robustness

Two mathematical techniques were applied to reduce the number of parameters used in defining the template. One technique was to apply kinematical constraints (i.e. fixed segment position when grippers attached). This constraint eliminated the gripping force parameter \( \kappa \). A second and more significant technique was to apply nondimensionalization. The process of nondimensionalization proved that the crawling system was not directly sensitive to all dimensional parameters but rather to a smaller number of nondimensional ratios. Nondimensionalization transformed the equation of motion (3-13), such that two dimensionless parameters (damping ratio \( \zeta \) and dimensionless gravity \( \mathcal{G} \)) replaced six dimensional modeling parameters: segment stiffness \( k \), segment damping \( c \), segment mass \( m \), peak muscle force \( A \), substrate inclination \( \theta \), and gravitation acceleration \( g \).

The template was able to generate caterpillar-like waves of gripping using only two nondimensional parameters suggesting that these parameters might be relevant to caterpillar locomotion. It may be of interest to measure these parameters across a wide range of caterpillar species and developmental stages, to determine whether caterpillars have converged to a particular set of values, or whether they are diverse. It is also possible that these two nondimensional parameters will be useful in developing scaling laws (Schmidt-Nielsen 1984) comparing locomotion for soft and rigid crawlers (e.g. comparing caterpillars to worms to snakes).

A compelling topic for future work is to explore whether this template can also describe crawling in earthworms (where “gripping” is generated from friction) or if a generalization of (3-15) might even describe crawling in flexible articulated-body animals such as snakes or lamprey (Dobrolyubov 1986).
3.6.2 Physical Robustness

Altogether, only seven dimensionless parameters influence simulated motion: two \((\zeta, \mathcal{G})\) appearing in the equation of motion, two \((\chi_{\text{init}}, \bar{I})\) in the boundary and initial conditions, and three more \((\Lambda_{\text{acc}}, \Lambda_{\text{vel}}, \Lambda_{\text{pos}})\) in the gripper control law.

Simulation results indicate minimal sensitivity to all dimensionless parameters, in the sense that successful gaits were observed over a wide range of the dimensionless parameter values tested.

Of the seven dimensionless parameters, the gripper parameters \((\Lambda_{\text{acc}}, \Lambda_{\text{vel}}, \Lambda_{\text{pos}})\) were somewhat more sensitive to failure than the others. This comparison is derived from a qualitative assessment of sensitivity: the ratio of the upper-to-lower bounds (ULB) of the parameter magnitudes that trigger failure (obtained from Table 3-2). In a qualitative sense, the system is less sensitive when the ULB is high. For the case of two parameters \((\zeta \text{ and } \bar{I})\), Table 2 reports that no upper bound was found even when the nominal parameter values were increased by an order of magnitude. In these cases, ULB might be said to “tend toward infinity.” For two other parameters \((\mathcal{G} \text{ and } \chi_{\text{init}})\), the range of allowable values includes zero. Since the lower magnitude bound is zero, the ULB for these cases might also be said to “tend toward infinity.” By contrast, the ULB metric for all three gripper parameters is finite. The metric is approximately 5 for \(\Lambda_{\text{acc}}\), 30 for \(\Lambda_{\text{vel}}\), and 4 for \(\Lambda_{\text{pos}}\). These parameters might be labeled \textit{moderately insensitive}, since ULB covers a wide (though not infinite) range of allowable values (though not an infinite range).

In the case of the four parameters to which crawling is robust \((\zeta, \mathcal{G}, \chi_{\text{init}}, \bar{I})\), it is relevant to note why failures occur in some extreme cases. Failure conditions can be triggered if the initial step length is too long \((\chi_{\text{init}} > 1)\) or if segment rest length is too short \((\bar{I} \rightarrow 0)\). For the
damping ratio, failures occur for underdamped cases \((\zeta < 1)\), a condition which indicates oscillatory response to external disturbances. In an underdamped system, oscillations can carry segments ahead or behind their neighbors, which is one of the defined failure conditions. Caterpillars found in nature are all significantly overdamped \((\zeta \gg 1)\), so such conditions are not biologically relevant, even if they are permitted by the template.

Physical robustness to damping ratio is indicative that the success of crawling depends little on the specific physical properties of caterpillar tissue. This observation suggests that there may be a wide range of tissue properties among different species (or developmental stages) of caterpillars, an unconfirmed hypothesis which might motivate biological experiments in the future. Failure to observe different tissue properties would not be sufficient to invalidate the template, but rather the presence of such variety could be explained by the sensitivity analysis shown here.

Perhaps the most interesting failure condition occurs for high values of dimensionless gravity \((\mathcal{G})\), where gravity can cause segments to drop past their neighbors during climbing if muscle forces are too weak to support free segments. The highest value of dimensionless gravity that occurred before failure (from Table 2) was 0.13, or approximately 1/8. A possible interpretation for this result can be obtained by realizing that unity \(\mathcal{G}\) implies that a muscle can support one segment statically against gravity. In nature, it has been observed that one segment can support the weight of the entire caterpillar body; in simulation, similarly, the muscles must be strong enough \((\mathcal{G} < 1/6)\) to support the entire six-segment body with margin remaining to accelerate the caterpillar against gravity.
The fact that the simulated model crawls successfully over a wide range of dimensionless gravity values suggests that gravity does not factor strongly into caterpillar locomotion. This result coupled with the observation that distance travelled is a function only of $\chi_{init}$ corroborates the experimental observation that caterpillar crawling kinematics are nearly identical for horizontal and vertical crawling (van Griethuijsen and Trimmer 2009).

### 3.6.3 Biological Robustness

Parameter sensitivities not resolved by mathematical or physical robustness must be mitigated by some other mechanism, at least for real animals to be viable under widely ranging environmental and developmental conditions. Though crawling success appears to be highly robust to all parameter values, it is relevant to carefully consider the gripper control parameters ($\Lambda_{acc}, \Lambda_{vel}, \Lambda_{pos}$), as the simulation did eventually fail when these parameters were varied over a wide enough range. In concept, biological robustness might shore up any remaining sensitivity of crawling success to these parameters. One example of biological robustness might be for caterpillars to supplement reflex-based control with a backup control strategy. For instance, a CPG-based backup controller might also be used, where the CPG relies on a timer to trigger a default motion pattern when the reflex-based controller fails. Biological robustness might also be achieved by adaptive learning, which would allow the gripper control parameters to be tuned were they to depart the allowable range (due to damage, aging, etc.) These hypotheses regarding biological robustness motivate compelling future research to identify mechanisms used by caterpillars to recover adaptively in scenarios of parameter uncertainty.

### 3.7 Conclusion

This chapter develops a mathematical model for soft-bodied crawling with reflex-based substrate gripping. The model was verified by comparing computer simulations to experimentally
acquired kinematics data. The key finding of the simulation study was that the model proved highly robust to parameter uncertainty. This robustness suggests that reflex-based substrate gripping is at least a viable strategy for generating locomotion in biological caterpillars. Its robustness further suggests that the model is a suitable basis for designing soft-bodied robots to operate in varied terrain.

The model developed in this chapter is in the form of a template, meaning that it is a drastic abstraction from the organism of study but may provide insight into an entire class of locomotion. Treating the caterpillar as a spring with the ability to shorten and lengthen via the action of muscles and crochets reveals the interactions that make up the bulk of the motion. Establishing a model for muscle/crochet coordination strengthens the notion of an environment skeleton, which has been proposed as an explanation for how soft animals are able to generate forward propulsion.

Robustness makes application of this control strategy to soft-bodied robots very compelling. It was suggested that any successful model of caterpillars would be relatively insensitive to a wide variety of environmental parameters because of the extensive changes that they undergo during their lifetimes. The robustness of the soft-body crawling template suggests it can be employed in robots with quite different mechanical properties provided they are equipped with suitable grippers....
4 Thesis Conclusion

4.1 Contributions

This thesis aims to advance the study of soft-bodied robotics by providing new control schemes and methods of analysis. A novel control scheme which produces a limit cycle was developed in Chapter 2. In the second topic (Chapter 3) a way of coordinating the actions of body segments in a soft bodied organism was proposed. It is hoped that this method might both inform design of soft caterpillar-inspired robots and extend to a whole class of similar systems. The future of soft-bodied crawling robots will benefit from this research by combining these two ideas in a manner that allows the natural dynamics of the system to provide stability over a sizable operating range but which has the ability to utilize controls producing a limit cycle to expand the region over which they move in predictable and controlled manner. During standard operation a tube-like soft robot mimicking M. sexta can reliably crawl using a scheme similar to that presented in Chapter 3. This region of stability may be sufficient for some search operations but one can readily imagine an instance when the environment might start to move and shift producing dynamics outside of the anticipated range. In this case having a second controller that is able to reject further disturbances could allow that robot to complete its mission.

As stated in the introduction, the major contributions of this thesis are:

(Chapter 2 – Development of Limit Cycles in a Class of Hybrid Systems Consisting of Two Linear Subsystems)

- Development of a procedure to generate a control system which produces a stable output of constant amplitude and frequency. By working with a hybrid system it was shown how two linear systems can be pieced together to create a system with a stable limit cycle
trajectory. Since the trajectory is a limit cycle it has a specific amplitude and frequency. It was further shown that the proposed method provides a way to alter these properties continuously. Prior research into the development of limit cycles in hybrid systems focused on showing their existence in a given specification. This chapter expanded from existence to showing how to design systems which had stable limit cycle trajectories. Limit cycles in soft robotics will allow designers to focus on other aspects of the robot and allow the controller to handle all of the unexpected disturbances that the world can throw at their creation.

- **Proving the stability of this system mathematically.** Rigor is sometimes viewed as unnecessary in engineering, but it is the author’s opinion that a solid foundation is important to the safe development of an idea. By using analysis to ensure the stability of the system presented this chapter allows future engineers to borrow the proof techniques and underpinnings of this procedure without worrying about when they are valid.

- **Verifying, through simulation, that the design procedure works.** As the Russian proverb goes: trust, but verify. Even with the mathematics securely evaluated it is important to present a graphical representation of those equations whenever possible. The simulation shown in this chapter gives someone unfamiliar with the mathematics a way of reassuring themselves that the method is sound. Adoption of this scheme is sped up by going through the design procedure in a concrete manner.
(Chapter 3 - Template for Robust Soft-Body Crawling with Reflex-Triggered Gripping)

- *Generation of a control system for a class of soft robots, specifically those employing grippers to attach to a substrate.* Caterpillars use grippers to keep their body firmly attached to the plant on which they are climbing. This forms the basis for what is known as the external skeleton. A mechanical system modeled after caterpillars might be able to accomplish all of the maneuvers that a real caterpillar is able to do. This chapter outlines a model (which is labeled a template because of its generality) and then goes about detailing how controls can be designed to allow this model to provide locomotion. While the grippers are useful to hold the animal onto the substrate this means that they need to be removed in order for the animal to move. The control system coordinates the motion to organize the lift-off and reattachment of the grippers. Such a coordinating system can be used in mechanical systems which want to borrow a similar gripping system because of the advantages it provides.

- *Characterization of the robustness of the template.* The surprising part of the model and control development is just how insensitive the system is to parameter selection. How much the caterpillar changes over the course of its life should have been an indication that this would be the case, but there is something remarkable in how readily the system can have its parameter space reduced. This stability is important to designers who are looking to develop soft robots which are operating in unpredictable environments. Changes in temperature can result in a different damping ratio, but the model shows that these are simply rejected by the very nature of the system.
4.2 Future Work

To finally achieve the promise of soft-bodied robotics is going to require further study and analysis, but if the visions can be achieved all of that work will be worth it. This thesis starts to point the way for development of control systems in soft-bodied robots but is just one way-point on a long road. Limit cycles have amazing promise and there is a lot to learn concerning how to extend the models presented here into the complicated world of soft, non-linear systems.

The limit cycles developed in Chapter 2 relied on a linear, single-DOF linear system. There is an obvious natural extension of this method to nonlinear systems that approximate the necessary linear properties in the vicinity of the nominal trajectory. This would give a smaller region of stability but would find more utility in the world of soft materials. Additionally, the proposal needs to be extended to multi-DOF systems to produce trajectories which coordinate the frequency and offsets among a great many state variables. This could take the form of finding a new higher-dimensional switching surface. If the proposed design method can be further refined to admit arbitrary numbers of state variables then it will allow for crawling by high-DOF continuum robots which can reject disturbances to achieve a stable gait over rough terrain. Even a basic extension to 4 state variables would permit the application of the proposed method to developing very efficient walking robots.

As for the template presented in Chapter 3, work remains to explore the model’s biological implications. For instance, the model might be investigated further to see if it can describe the generation of different gaits (crawling vs. inching) in caterpillars. Just as with the SLIP model this template has the ability to speak to biologically relevant questions such as what role caterpillar geometry is constrained during crawling to prevent buckling and how scaling laws account for the needed muscle strength.
4.3 Technical Impact

An example of “search and rescue in a collapsed building” was offered as being a compelling reason to study the world of soft robotics. In order to understand how to go about developing soft robots for this purpose it is necessary to think about the challenges that such circumstances present. Soft-bodied rescue robots will not have a never ending power source so they need to be efficient. The world is full of uneven terrain and disturbances so any mobile robot will have to be able to deal with this. To get to where they need to be in a collapsed building these robots will have to maneuver through a very complex and unpredictable environment full of constricted passages. A rigid system might get hung up on such surroundings but soft systems could slip through by allowing their body to deform. This deformability is an asset and a liability. To properly take advantage of their softness methods to make motion more predictable need to be developed. All of these complications need to be addressed before soft robots will be ready for prime time.

The technical material presented in this thesis points the way toward establishing sustained, efficient, robust locomotion. Efficiency and stability are hallmarks of limit cycles so the work presented here on this topic provides a new formulation of control which may be applicable to future soft robots. Robots crawling in the perilous environments present after a disaster will be able to perform their jobs more effectively with control systems which reject the disturbances that may be present in such an environment. Robustness, as shown in chapter 3, will allow these rescue-bots to be constructed in a manner which is less concerned with exactness. Tolerance is a good measure of cost in producing objects, so systems which are able to operate over a wide range of parameters allow for looser tolerances and hence are cheaper. Out in the real world this robustness means that poor measurement from a sensor will not be catastrophic to
the successful completion of the appointed task. There is still a lot of work to be done to adapt these schemes to real soft systems, but perhaps this is at least a small step in that direction. Conquering the fundamentals of these systems will not solely be the product of profound human invention but will follow from observing natural organisms and finding ways to duplicate their best aspects. Soft robots will be able to perform tasks that are currently infeasible for rigid robots and have the potential to truly benefit humanity. Future study of such systems is important because of the advantages that they afford a designer.

The initial impetus of Chapter 3 was to extend the curvature derivative control laws of (Date and Takita 2007) to the world of caterpillars. What sets caterpillars apart from snakes is the presence of the crochets which allow the caterpillar to produce on-demand drastic changes in localized friction. This allows the caterpillar to directly ascent vertical surfaces with ease, reducing route planning and energy expenditure. Passive engagement of the crochets allows the caterpillar to maintain its body position without having to activate muscles; an immediate energy savings. When traversing complex terrain it is beneficial to be able to affix firmly to the ground as this provides a good body position reference to aid in continuation of locomotion. Future soft-bodied crawling robots utilizing crochets as part of their structure will have these advantages.

Locomotion which is the product of entraining a trajectory to a limit cycle is very popular in the natural world. We humans store energy in our pendulous arms during one part of our gait and release this energy at another point to power through the cycle. Such methods are able to stabilize in the presence of disturbances and capture efficiencies which may be lost in classical control designs. Instead of having an active controller constantly making adjustments a limit cycle controller may be able to allow the system to naturally return to the desired path. The limit cycle investigation in Chapter 2 may eventually be extended to multiple dimensions or applied
locally to a higher order system to grant some of its benefits to locomotion. Since the caterpillar of Chapter 3 relies only on information about two segments it may be possible for the motion of the segments to be made even more robust by introducing this additional control law. Applying the idea of limit cycles to the already robust crawling of the caterpillar borrows from the natural world and then takes it one step further.

Combining the ideas from Chapters 2 and 3 might also come in the form of a reintroduction of the CPG. While the role of the CPG has been reduced and refine there is a lot of experimental evidence that it is able to stabilize motion in a large class of mobile robots (Ayers and Witting 2006). A CPG or similar timing structure which is stabilized to a limit cycle would be able to keep a steady locomotive rhythm. The investigation in Chapter 2 shows how this rhythm can be modified continuously to produce any desired frequency. Reflex control from Chapter 3 could inform the selection of the frequency for the timer. This combination creates a hierarchy of control that is able to use both global and local stabilization.

Soft robots might not revolutionize the factory but they will fundamentally change what is meant by robot. The world is inhospitable to large metal behemoths but has proven to be relatively accommodating to innocuous, pliable tubes of goo. Robots which are similarly malleable will be able to distort themselves to fit through cracks and crevices which would not be traversable by their hard counterparts. Archeologists could use soft robots to explore the ancient world by having them crawl into the subterranean ruins instead of having to clear off millennia of dust. Biologists might use caterpillar-robots to investigate parts of the rainforest which are inaccessible to conventional robots or to burrow into tunnels where rigid robots simple cannot tread. One of these devices might one day squirm its way through rubble to deliver life
sustaining water to victims trapped in a collapsed building. This is the next revolution in robotics.
Appendix – Background Mathematics

Besides the biological connection that runs through this thesis there is a common tie in the implementation of the control. Hybrid systems are a new and developing area of study that has insight to offer to previously unsolved problems and which generates its own host of open questions. Systems which were previously too difficult to model can be realized within a hybrid context and systems that were once not able to be controlled now can be.

A.1.1 Control of Linear Systems

Both of the topics investigated in this thesis work with linear systems so a small introduction to such systems is provided. This area of study can be extended into years of learning, but the basics are readily understood and provide a good enough foundation to comprehend what is happening throughout this thesis. Any additional information needed on linear systems can be found in either a good linear algebra book or in a linear controls textbook.

Linear systems are those which obey principals of superposition and scaling. Such systems are commonly used because of the wealth of knowledge already available developed for them. A common general class of linear systems is the second order system. Depending on how the variables are specified this exact same equation can be used to model a wide variety of phenomena.

\[
 m\ddot{x} + \mu \dot{x} + kx = 0 \quad (A-1)
\]

Here the order is used to describe the highest degree derivative in the equation. Since this system scales and behaves with superposition it is linear. This behavior is shown in equation (A-2).
Linear systems do a good job at modeling a wide variety of physical phenomena. The equation in (A-1), for example, can be used to model the behavior of a spring-mass-damper system. In that case \( m \) is mass, \( \mu \) is the coefficient of damping, and \( k \) is the stiffness of the spring. The state variable \( x \) in this mechanical system is the displacement of the mass. Or, it can model an electrical circuit response by setting \( m \) to the inductance, \( \mu \) to the resistance, and \( k \) to the reciprocal of capacitance. This establishes an ordinary differential equation describing the value of current. This versatility makes linear systems an important area of modeling worthy of investigation.

One typical way of representing linear systems in controls theory is called state-space representation. In state-space each state variable is given its own linear equation, the collection of which is then compiled into a system of equations. In order for this to be accomplished the system of second order and higher equations is broken down into separate first order systems. For example equation (A-1) can be rewritten as a system of first order linear equations.

\[
H(x(t)) = m \frac{d^2}{dt^2}(x(t)) + \mu \frac{d}{dt}(x(t)) + kx(t) \quad (A-2)
\]

\[
H(\alpha x_1 + \beta x_2) = m \frac{d^2}{dt^2}(\alpha x_1 + \beta x_2) + \mu \frac{d}{dt}(\alpha x_1 + \beta x_2) + k(\alpha x_1 + \beta x_2)
\]

\[
H(\alpha x_1 + \beta x_2) = m\alpha \ddot{x}_1 + m\beta \ddot{x}_2 + \mu \alpha \dot{x}_1 + \mu \beta \dot{x}_2 + k\alpha x_1 + k\beta x_2
\]

\[
H(\alpha x_1 + \beta x_2) = \alpha H(x_1) + \beta H(x_2)
\]
We can make this expression compact by reducing the vectors and matrices to variables.

\[ \dot{x} = Ax \]  \hspace{1cm} (A-4)

The behavior of such systems classified by the eigenvalues of the matrix \( A \). The system is said to be unstable if the real parts of any eigenvalue is positive, and stable if it is not unstable. This classification is made based upon what occurs when the system is disrupted. When a system has positive eigenvalues the system response will increase without bound when a stimulus is applied. In a similar manner a stable system will have a bounded response to a given stimulus. In a controllable system \( K \) can be selected such that the eigenvalues are whatever the designer desires (with the restriction that complex eigenvalues be generated as conjugate pairs). If there are complex eigenvalues then the system will oscillate, otherwise the response will be monotonic. The marginal case where the eigenvalues are purely imaginary elicits a response which oscillates at constant amplitude.

In order to better understand what is meant by stable and unstable, oscillatory and monotonic we can look at the solution to the ODE in (A-3) for various parameter configurations. Since there are two state variables it makes sense to plot one against the other in what is called a phase plot. In the case of mechanical systems this is plotting position versus velocity. The solution to ODEs requires knowing initial conditions so it is sometimes more informative to populate the entire region of interest with vectors indicating the direction that a curve passing through that point is directed. A function which assigns to each point in a manifold an element of the tangent space at that point is called a vector field. The second order system given by (A-1) exists in \( \mathbb{R}^2 \) and has a vector field given by (A-3) to denote how a curve passing through a given point will change in each of the state variables with respect to time.
The eigenvalues of the state matrix of (A-3) are given as solutions to a polynomial equation.

\[
eig = \left\{ \lambda \left| \det \left( \begin{pmatrix} 0 & \frac{1}{m} \\ -\frac{k}{m} & -\frac{\mu}{m} - \lambda \end{pmatrix} - \lambda I \right) = 0 \right. \right\} \tag{A-5}
\]

\[
\det \left( \begin{pmatrix} 0 - \lambda & \frac{1}{m} \\ -\frac{k}{m} & -\frac{\mu}{m} - \lambda \end{pmatrix} \right) = 0
\]

\[
(-\lambda) \left( -\frac{\mu}{m} - \lambda \right) - \left( -\frac{k}{m} \right) = 0
\]

\[
\lambda^2 + \lambda \left( \frac{\mu}{m} \right) + \left( \frac{k}{m} \right) = 0
\]

\[
\lambda = \frac{-\left( \frac{\mu}{m} \right) \pm \sqrt{\left( \frac{\mu}{m} \right)^2 - 4 \left( \frac{k}{m} \right)}}{2}
\]

For this exploration of system response consider the following values of \( m, \mu, \) and \( k \) which result in the given eigenvalues.

<table>
<thead>
<tr>
<th></th>
<th>( m )</th>
<th>( \mu )</th>
<th>( k )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable, real</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Unstable, real</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Stable, complex</td>
<td>1</td>
<td>-0.2</td>
<td>-2</td>
<td>-0.1\pm0.14i</td>
</tr>
<tr>
<td>Unstable, complex</td>
<td>1</td>
<td>0.2</td>
<td>-2</td>
<td>0.1\pm0.14i</td>
</tr>
<tr>
<td>Stable, imaginary</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>\pm1i</td>
</tr>
</tbody>
</table>

The phase plots for these systems, along with an example trajectory, are given in figure Figure A-1 - Phase portraits of systems exhibiting various categories of eigenvalues. Phase plots show the way a system will “flow” as it evolves. Typically they show position along the x-axis.
and velocity along the y-axis. Points along a trajectory, then, are the state of the system at a given time in terms of its position and velocity.

- Stable, real
- Unstable, Real
- Stable, complex
- Unstable, complex
Stable, imaginary

Figure A-1- Phase portraits of systems exhibiting various categories of eigenvalues

In typical feedback control applications the system is broken down into two subsystems: the plant and the controller. The plant is what is being controlled. It has an associated model that describes how it behaves under certain influences. The controller provides influences to the plant to give a desired outcome. A controller also has to be modeled, though in modern digital control this distinction is blurred a bit as the controller is frequently just a computer able to influence the plant far faster than the changes that the plant exhibits. Feedback comes into play through the fact that the controller is constantly monitoring the plant and adjusting the influence it imparts based on those measurements.

Addition is a linear operation so the sum of two linear systems over the same variables is also linear. This is important because it allows for a simple method of envisioning control of a linear system as the sum of the system output and the control output. There are many methods for controlling linear systems, but the method given in (A-6), called full-state feedback, takes advantage of this property of linearity without introducing any more variables than present in the plant model.
Where \( A \) is a matrix defining the behavior of the system, \( B \) is a matrix specifying the state variables that can be controlled and the gain inherent to their actuator, and \( K \) is a matrix of control coefficients which are selected to give the controlled system the desired behavior. It may also not be possible to monitor all of the state variables as they evolve in time. To account for the ability or inability to observe certain state variables an output vector equation is defined to detail which are sensed and how the control affects the detected values.

\[
y(t) = Cx(t) + Du(t) \tag{A-7}
\]

Where \( C \) categorizes the ability to sense each state variable (and their interaction between each other) and the gain associated with its observation and \( D \) accounts for the influence of the control on the observed variable.

In (A-6) the linear system \( \dot{x}(t) = Ax(t) \) (called the “plant”) is coupled to a control scheme \( \dot{x}(t) = Bu(t) \). In order for the resultant to be linear the two parts need to be linear over the same variables. To achieve this the control equation \( u(t) = Kx(t) \) is substituted. This substitution makes both the plant and the controller linear over the same variables. The system behavior can now be evaluated using the tools available for linear systems.

In general the control of systems focuses on how to stabilize the system in a manner such as that seen in the stable (complex or real) phase plots of figure Figure A-1. Phase portraits of systems exhibiting various categories of eigenvalues. For example the linear system that is being investigated here can be used to model the cruise control in a car. The desired action of cruise
control is to maintain a given set point (which can be mapped to the origin). Stable systems will follow a trajectory towards the origin regardless of where they start. If we assume that the system is stable and that it is operating at steady state then we can think of a disturbance as something which causes the system to achieve a state outside of the origin. This disturbed state can be described by a position and velocity which maps to some point in the phase plot. After the disturbance is removed the system will follow the trajectory from the initial (disturbed point) back to the origin according to the governing equations.

A.1.2 Hybrid Systems

Linear systems are a very studied area of control, but there are numerous others. This thesis works to fuse the desirable characteristics of linear systems with the wider range non-linear control system. In this case linear systems will be subordinate to an overarching non-linear control scheme. This will give the system the ability to exhibit properties which are not found in linear systems without losing all of their benefits.

While a lot of controls research focuses on this sort of stabilizing influence part One of this thesis develops a different form of stability whereby the trajectory achieves some closed orbit. Such a trajectory is called a limit cycle and it has many desirable properties. Consider the case of a human walking, with a leg modeled as a pendulum. After applying linearization a system such as (A-3) can be used to describe the behavior of the model. We would not want the swinging of the leg to stop, but rather ideally it would continue to swing back and forth indefinitely. This desired motion would produce a phase portrait like the stable, imaginary plot in figure Figure A-1- Phase portraits of systems exhibiting various categories of eigenvalues. It should be noted, however, that the trajectory in that plot is stable in some concentric orbit depending on the initial condition. In the case of the stable (complex or real) system the initial
condition determined the trajectory as well, but regardless of what path it followed the system ended up at the origin. Disturbances of the stable orbit have the effect of moving the system to some other point on the phase plane which totally changes the orbit. In the case of walking this would mean that instead of having a steady gait every bump in the terrain would fundamentally change how far the leg swings. This is where a limit cycle comes into play.

Stable limit cycles are stable orbits in the phase plane which have some region outside of the orbit where the trajectory is directed back towards the orbit. Whereas the stable, imaginary system would not regain its walking gait, a system which followed a limit cycle would return to that limit cycle after the disturbance was removed. It should be obvious, then, why this idea of stability has applicability to systems which move. However, limit cycles are not a trajectory that can be achieved by linear systems. What part One of this thesis does is produce a construct where the tools of linear systems can be used to produce limit cycles.

Hybrid systems allow a designer to choose control schemes based on what is being sensed. These control selections can result in rapid changes that are not allowed in solutions to the typical ordinary differential equations used to model systems. However, as seen in the following example, reality is sometimes modeled better by allowing these discontinuities.

Consider a ball. Apply abstraction to arrive at the ideal ball; the essence of ball. Try to identify what is important and eschew everything else. It probably has a mass. Now imagine this ball is released in the presence of a uniform gravitational field. It is well understood what will happen: the ball will accelerate through the field. This is because of the single most fundamental law of physics: Newton’s Second Law of Motion.

\[ F = ma \]  (A-8)
The force exerted by the gravitational field is proportional to mass and the magnitude of the field. Rewriting (A-8) with a generic position variable \( x \) gives an ordinary (ordinary) differential equation.

\[
\ddot{x} = g \tag{A-9}
\]

Where \( \ddot{x} \) is the second derivative of position with respect to time and \( g \) is the intensity of the gravitational field. It is relatively straight forward to solve this equation for position. Doing so gives:

\[
x(t) = x_0 + v_0 t + \frac{1}{2} gt^2 \tag{A-10}
\]

Where \( x_0 \) and \( v_0 \) are a couple of constants of integration that represent the initial position and velocity of the ball. Typical experiences involving balls being dropped involve an obvious and convenient way to select where the zero of the position variable is. In this manner specify that \( x = 0 \) at the floor; that the ball is being suspended at some positive height \( x_0 \), and that the gravitational field acts in the negative direction. Release the ball. As the ball falls towards the floor take a moment to consider what is about to happen. Backed by a wealth of casual experience with dropping things anticipate what will occur. How much time it will take for the ball will reach position \( x = 0 \) can be found using equation (A-10).

\[
t(x = 0) = \sqrt{\frac{2x_0}{g}} \tag{A-11}
\]

Place yourself in this moment as the ball reaches \( x = 0 \). Push time forward just a little bit more, past that critical point. In your mind, where is the ball now? Of course, it has seamlessly passed through the floor as it continues to accelerate forever into the void. Protests are not heard
as the indomitable force of (A-8) guides all of reality forward. “There must be something more to this,” you exclaim! In all of the previous experiences the ball rebounded.

These prior balls interacted with the floor and through exchange of energy between the ball and the floor the ball changes direction and rebounded at some speed less than it had previously. Each atom in the ball and the floor was governed by various equations but the net result was the ball rebounding and some new speed. If we assume that all of the meaningful interactions are classical in nature then the collision could be written as a large collection of continuous equations. It would be impossible to solve and intermediate results would likely do nothing to elucidate what was occurring. Engineering is about results. So, borrowing knowledge from prior experiences a model can be constructed which matches exactly what has been observed.

![Diagram](image)

**Figure A-2- Hybrid system of a ball in flight**

By eliminating the complications of the collision interaction and replacing them with a simple rule a system is arrived at that matches experience. This diagram is one of the simplest possible hybrid systems but it provides a good illustrative example to understand the nomenclature and power of this type of formulation.
Hybrid systems consist of a collection of two types of variables: continuous and discrete. The continuous variables (typically referred to as $X = \{x_1, ..., x_n\}$) are those variables which evolve continuously with respect to the independent variable (usually time). How the continuous state develops is governed by the continuous equation which is specified by the active discrete state. The active discrete state is specified by the discrete variables (typically $Q = \{q_1, ..., q_m\}$).

If we specify the dimension of the continuous state space as $n$ then the entire space of the system can be described by a vector field $f: Q \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Transitions between discrete states are limited to occurring along defined edges and are controlled by guarding conditions. The set of all edges $E = \{(u, v)\}$ specifies all of the pairs of discrete states between which transitions can occur. Since not all transitions are reversible the pairs in that make up the edge set are ordered from the ‘current’ to the ‘next’ discrete state. Assigned to each edge is a guard condition $g$ which determines when a transition can occur and reset map $r$ which may alter the value of the continuous variables discreetly and discontinuously. Finally there is a set $Init$ which consists of all $(n + 1)$-tuples of conditions that the system may start with. In the case of the ball being dropped $Init$ would be the infinite set $\{(FLY, x, \dot{x})|x > 0, \dot{x} \in \mathbb{R}\}$.

Since not all variables may be sensed or reported the model can be extended to include restrictions on what the input and output of each discrete state can be. This is similar to the $B$ and $C$ matrices typically used in linear state space models of continuous systems. The collection of transitions can be viewed as an adjacency matrix for the graph with discrete states as nodes. In the same way that $B$ and $C$ can be used to determine controllability and observability, the adjacency matrix can be used to find discrete states which are not reachable or which are inescapable.
Hybrid systems are a natural extension of control theory, especially in the modern digital context. The flipping of a switch is necessarily a binary change, so controls that incorporate switches need the ability to discontinuously change. These discontinuous and nonlinear changes are also important because they present nonlinearities which can be leveraged for novel system control. In the case of linear controls there are only two possible outcomes: stable and unstable. Nonlinear systems on the other hand have an entire menagerie of solutions such as solitons, limit cycles, attractors, and fractals. Solutions exhibiting these exotic properties are useful in a variety of applications.
B Bibliography


